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ISIT 2016

Information Decomposition on Structured Space

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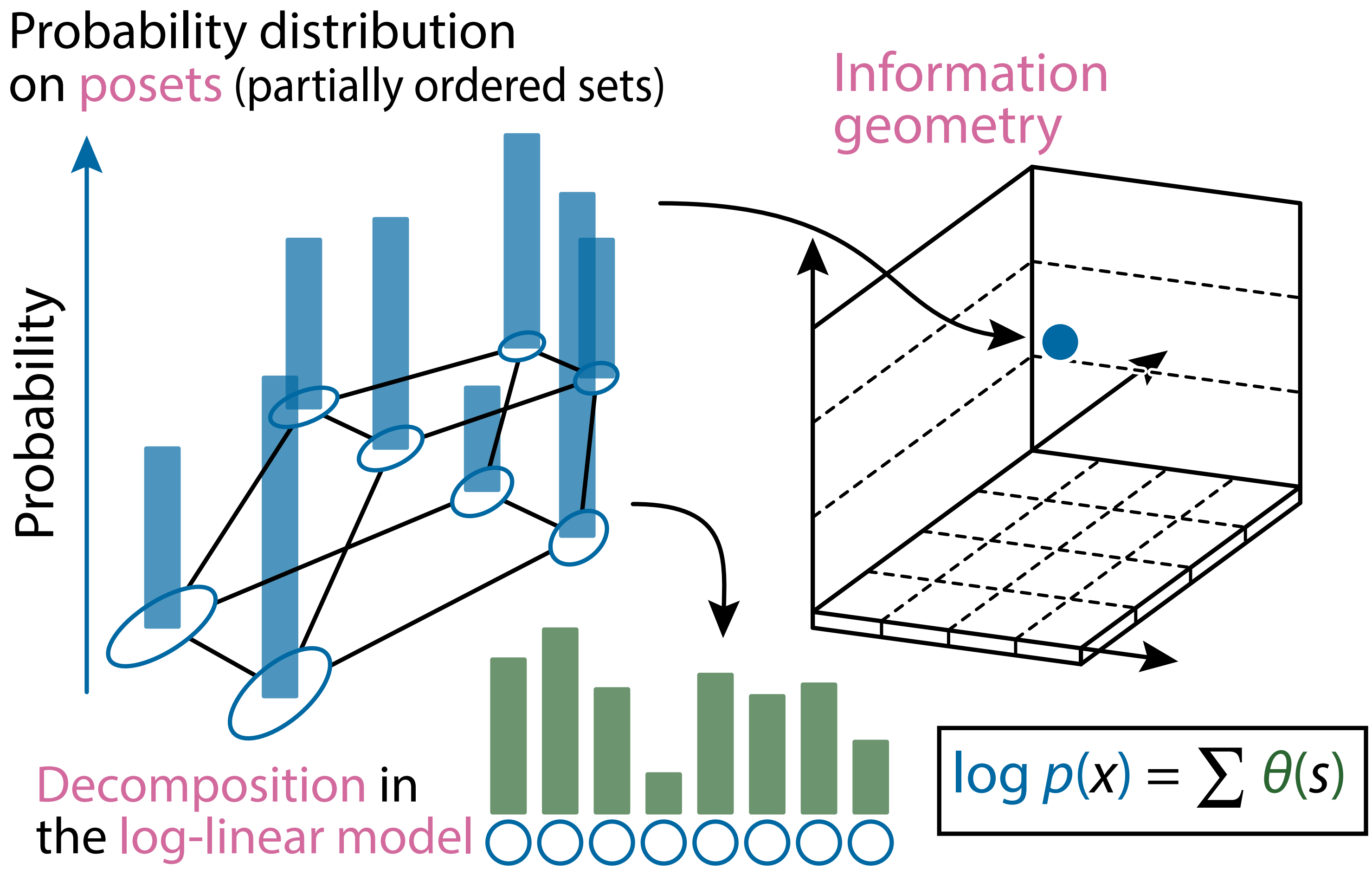
Contributions

- We build information geometry for posets (partially ordered sets)
  - Decomposition of KL divergence
- Key observations:
  - θ-coordinate → principal ideals (lower sets) → p-coordinate
    - θ-coordinate: coefficients of a log-linear model
    - p-coordinate: probabilities
  - p-coordinate → principal filters (upper sets) → η-coordinate
    - η-coordinate: frequencies (sufficient statistics)
- Code: `https://git.io/decomp`

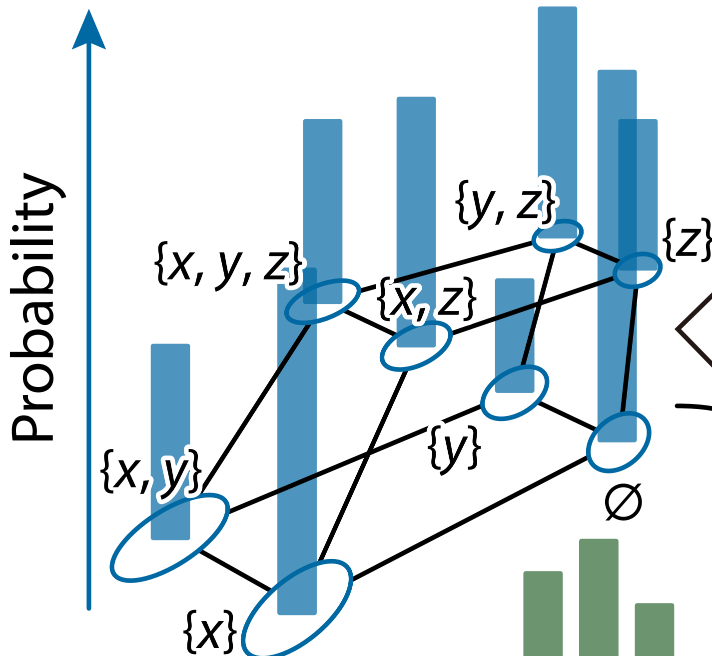
Summary



Probability distribution
on posets (partially ordered sets)
Information
geometry
Probability
Decomposition in
the log-linear model
$\log p(x) = \sum \theta(s)$

Summary

Probability distribution
on posets (partially ordered sets)

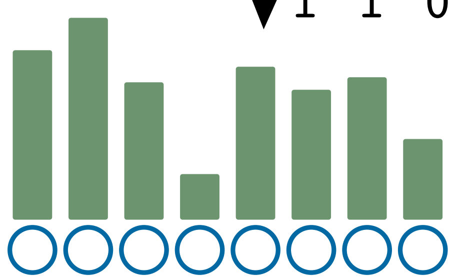


x y z (e.g. Neurons, SNPs, ...)

x	y	z	
0	0	1	$\cdots$
1	0	0	$\cdots$
1	1	1	$\cdots$
0	0	0	$\cdots$
1	1	0	$\cdots$
0	1	1	$\cdots$
1	0	1	$\cdots$
1	0	1	$\cdots$
1	0	1	$\cdots$
1	1	0	$\cdots$

Numerical score (KL divergence) and the p-value for higher-order intractions

Decomposition in
the log-linear model

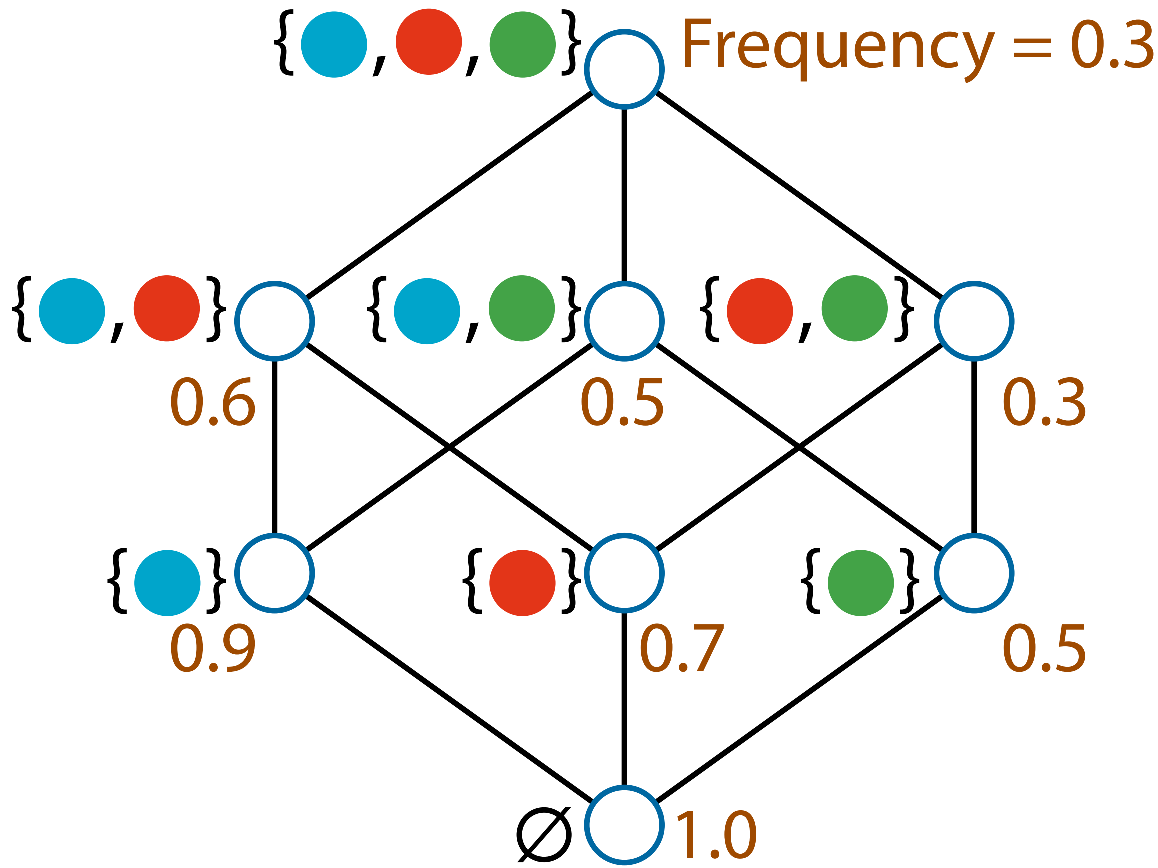


$$\log p(x) = \sum \theta(s)$$

Transaction database

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

Itemset lattice



Transaction database

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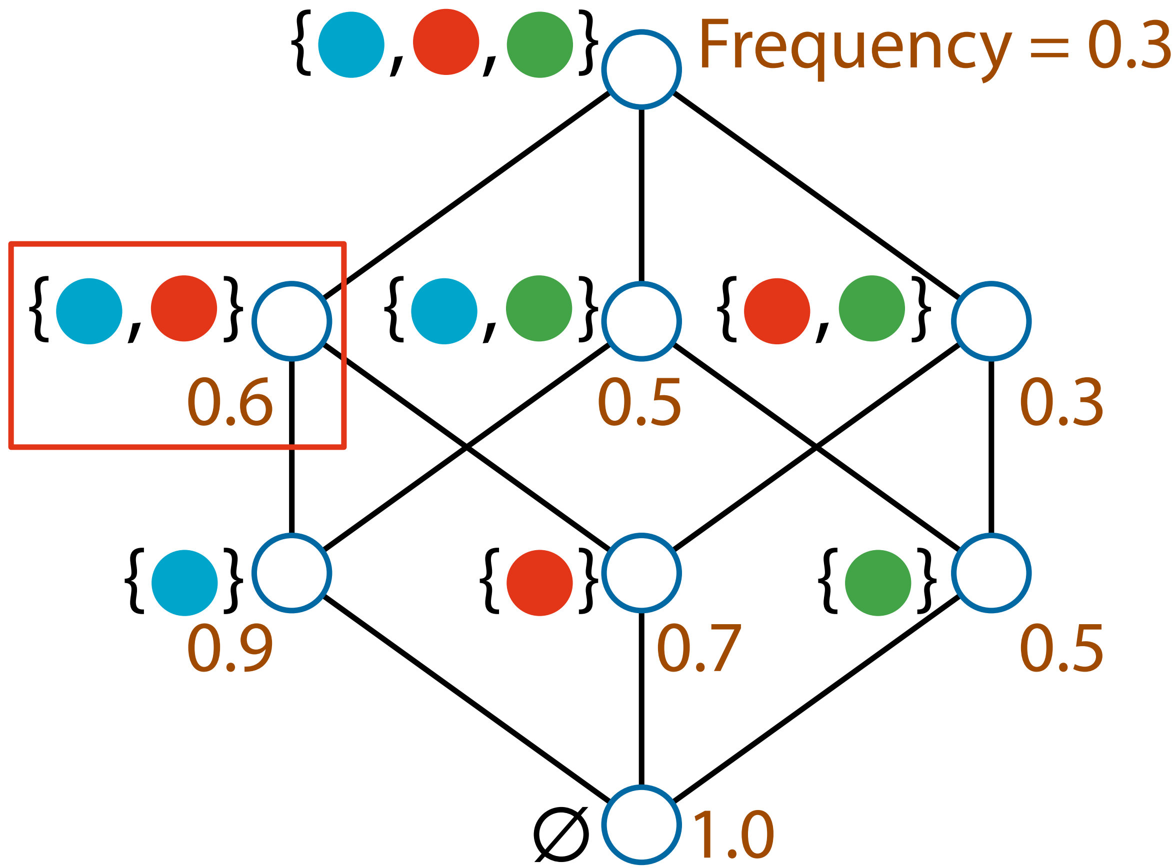
ID 9:	1	0	0
ID10:	0	1	0

Itemset lattice



Transaction database

ID 1:	1	1	0
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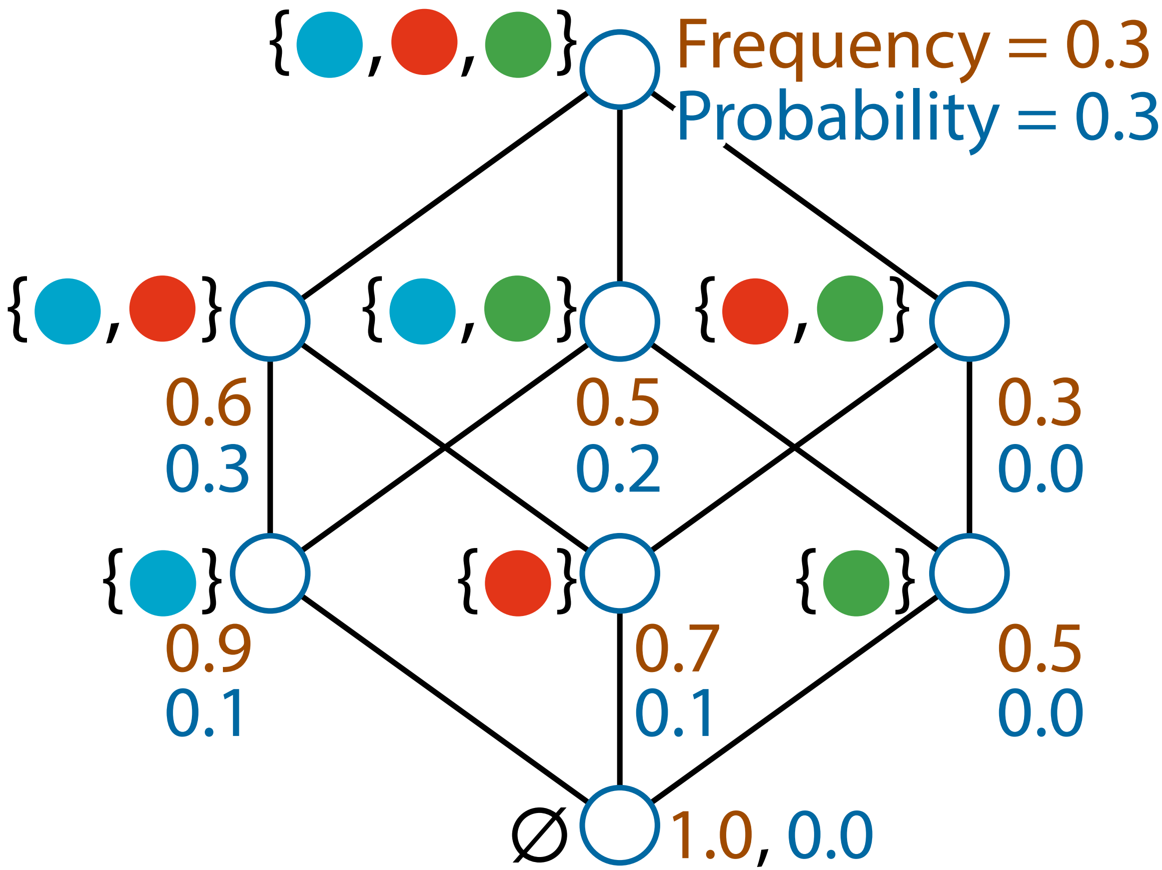
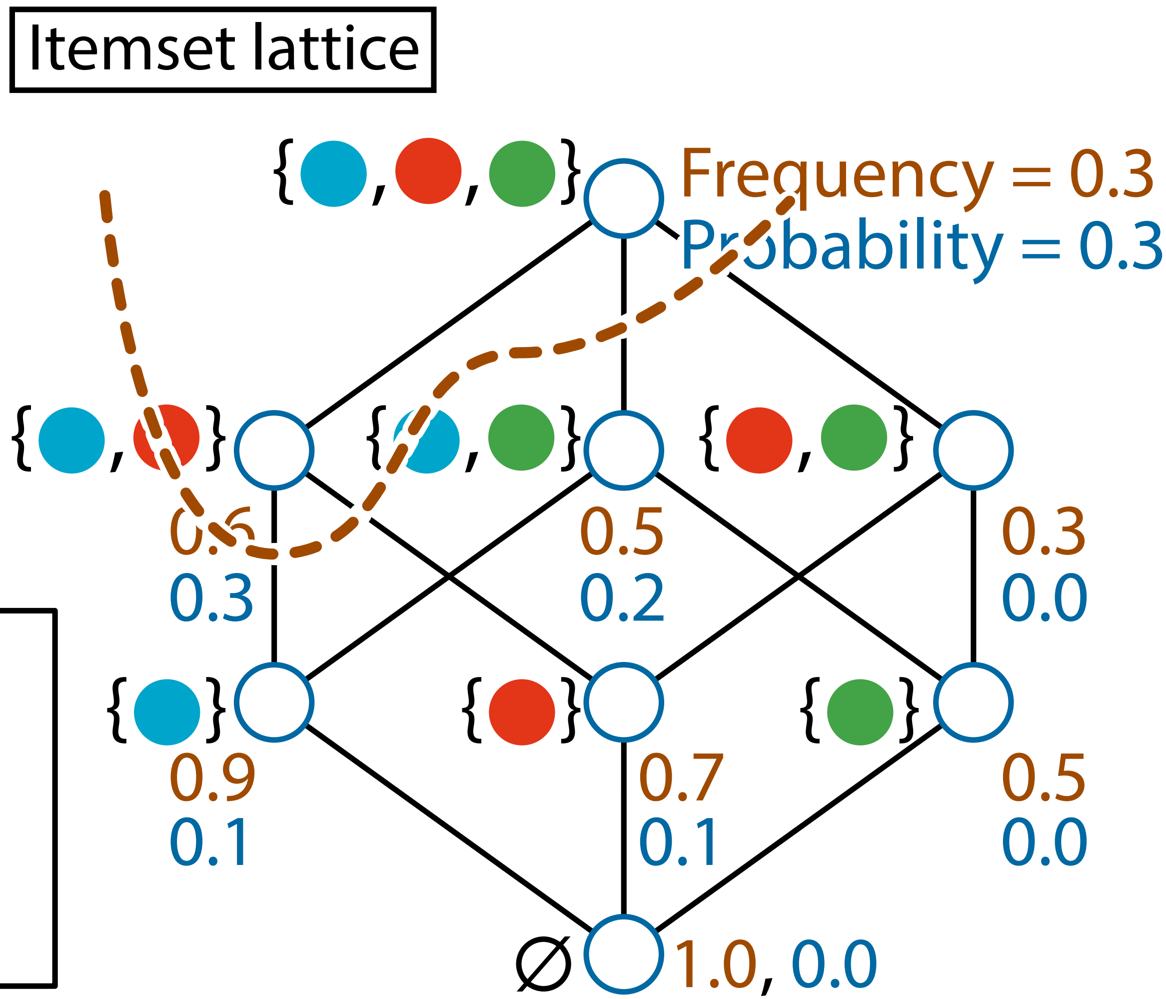
Itemset lattice



Upward = Pattern mining



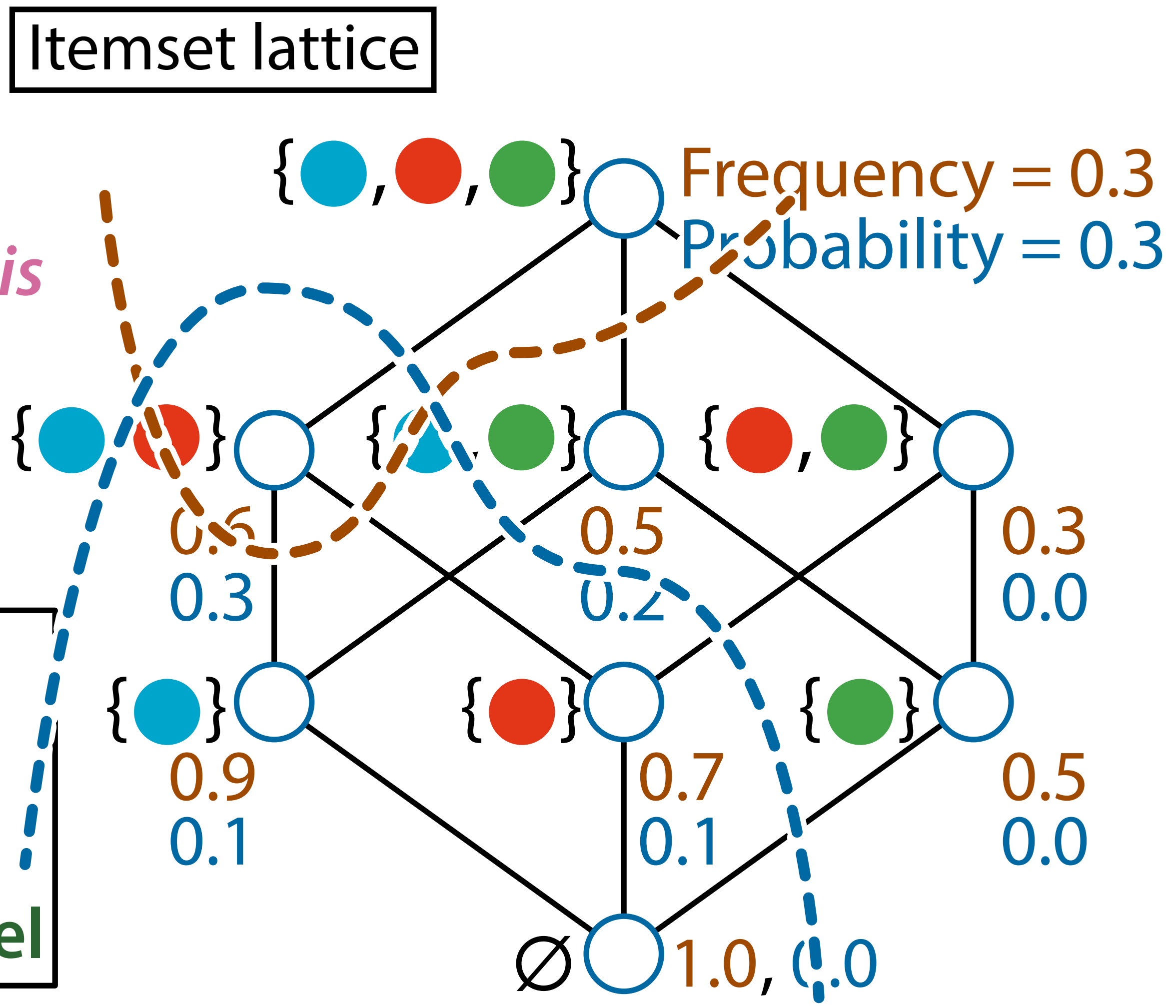
η: Frequency

p: Probability

$\eta(\{\bullet,\bullet\}) = p(\{\bullet,\bullet\}) + p(\{\bullet,\bullet,\bullet\})$

Upward = Pattern mining

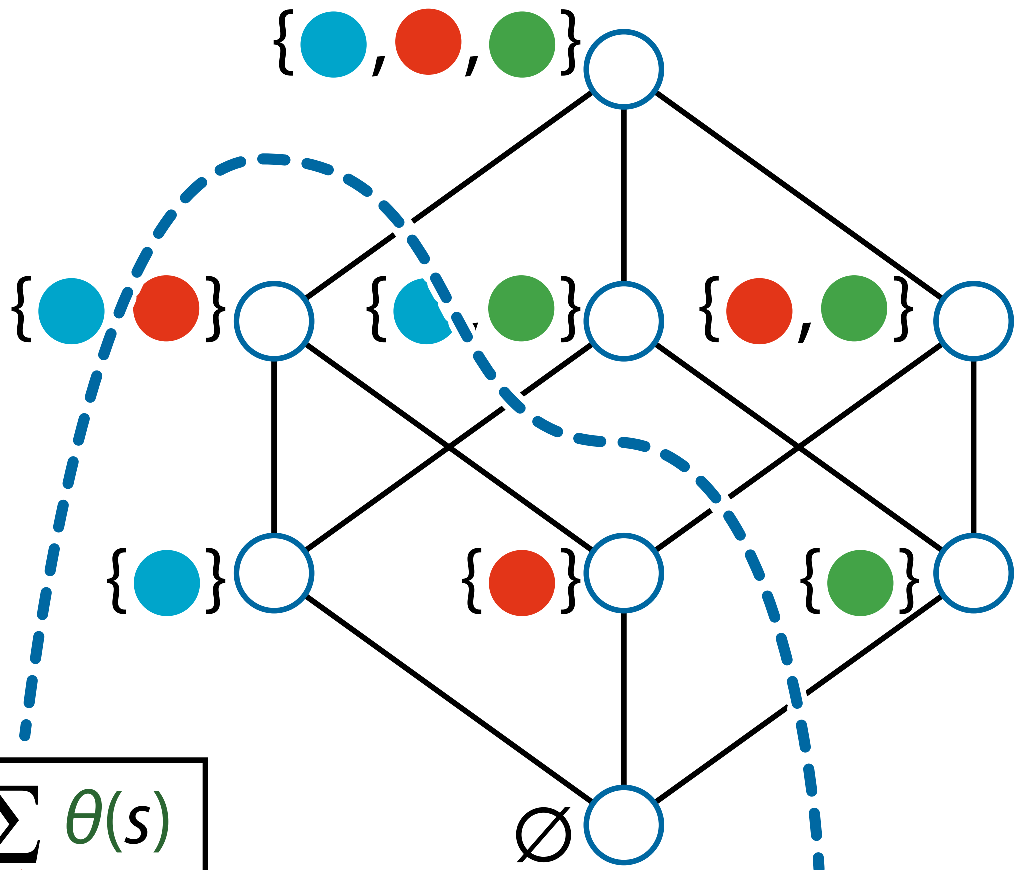
Downward = Log-linear analysis



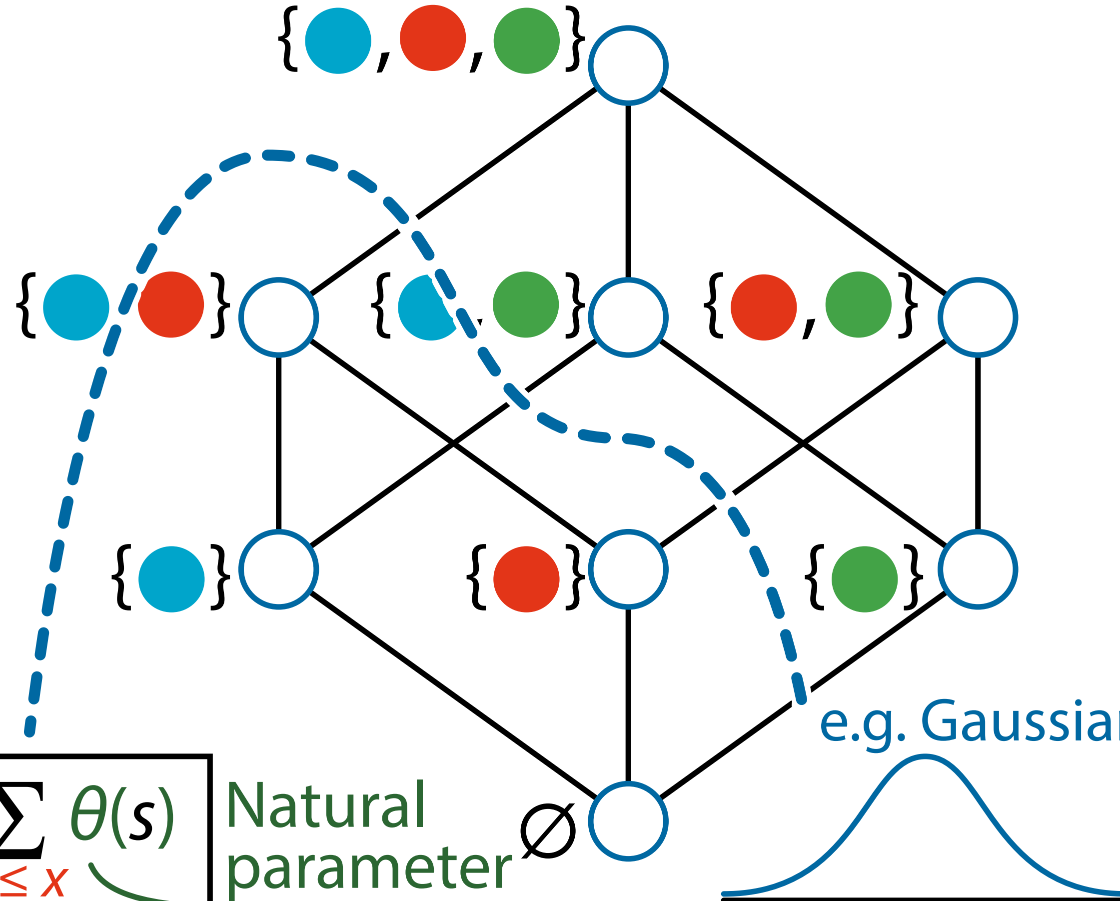
η: Frequency
p: Probability
θ: Coefficient of **log-linear model**

$$\eta(\{{\color{cyan}\bullet},{\color{red}\bullet}\}) = p(\{{\color{cyan}\bullet},{\color{red}\bullet}\}) + p(\{{\color{cyan}\bullet},{\color{red}\bullet},{\color{green}\bullet}\})$$

$$\log p(\{{\color{cyan}\bullet},{\color{red}\bullet}\}) = \theta(\{{\color{cyan}\bullet},{\color{red}\bullet}\}) + \theta(\{{\color{cyan}\bullet}\}) + \theta(\{{\color{red}\bullet}\}) + \theta(\emptyset)$$



$$\log p(x) = \sum_{s \le x} \theta(s)$$



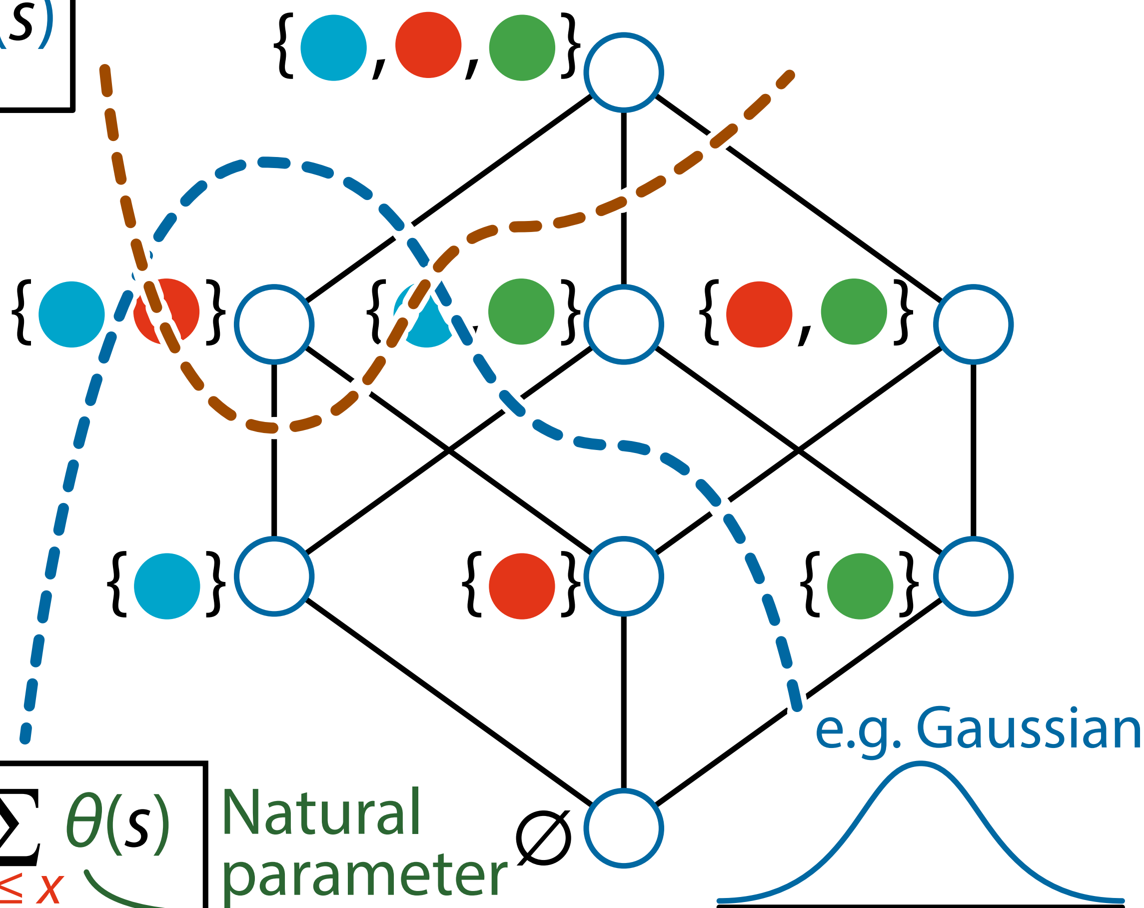
$$\log p(x) = \sum_{s \le x} \theta(s)$$

Exponential family: $p(x) = \exp\left(\sum \theta(s) F_s(x) - \psi(\theta) \right)$

$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\eta(x) = \mathbb{E}[\, F_x(s) \,]$$

Sufficient statistics of exponential family

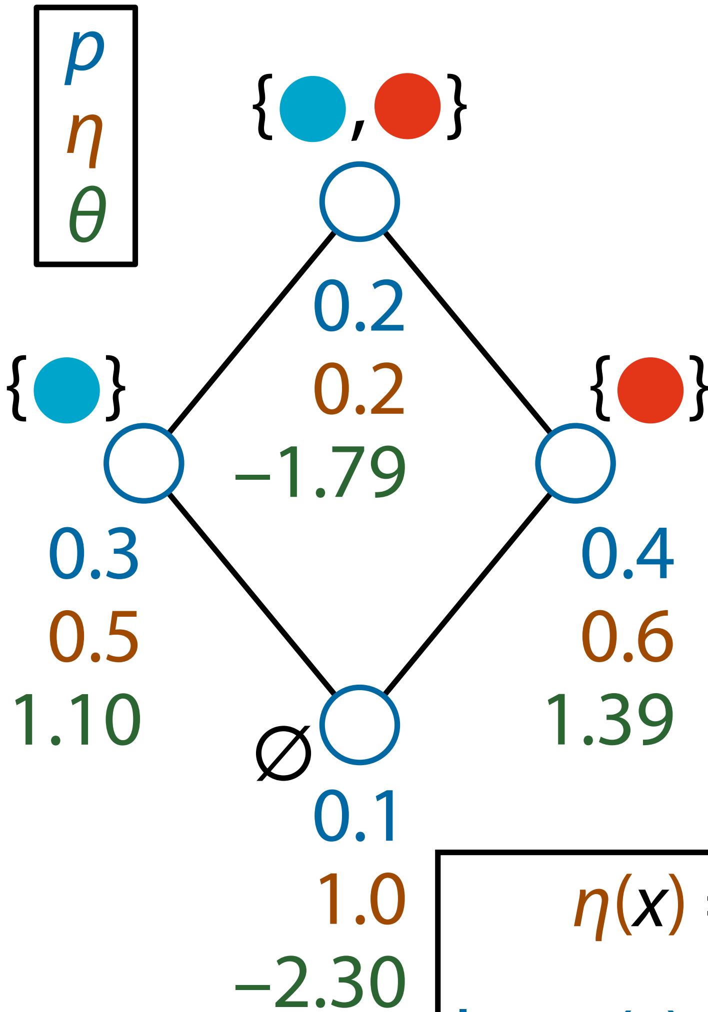


$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Natural parameter

Exponential family: $p(x) = \exp\left(\sum \theta(s) F_s(x) - \psi(\theta) \right)$

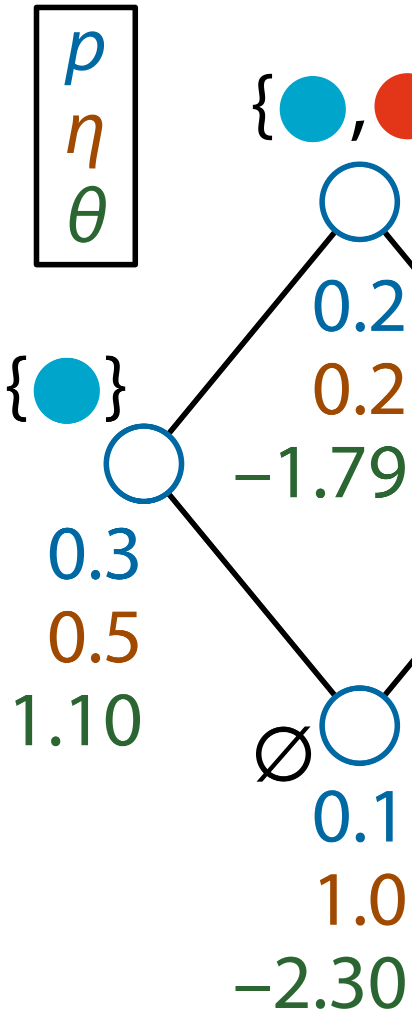
Triple for each node

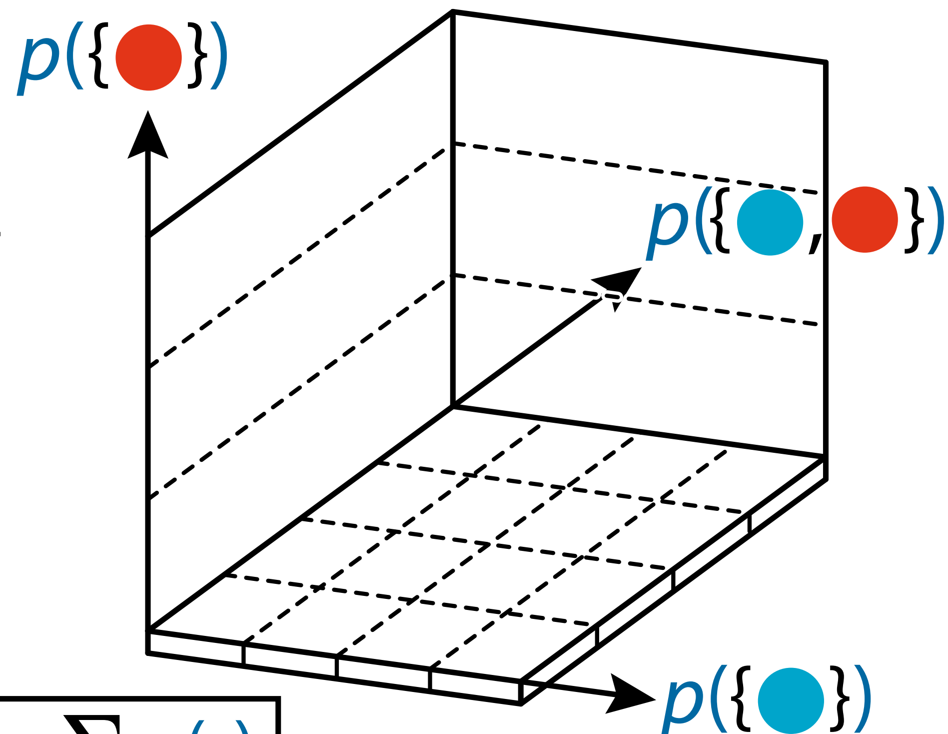


$$\eta(x) = \sum_{s \geq x} p(s)$$

$$\log p(x) = \sum_{s \leq x} \theta(s)$$

Triple for each node





$$\eta(x) = \sum_{s \geq x} p(s)$$

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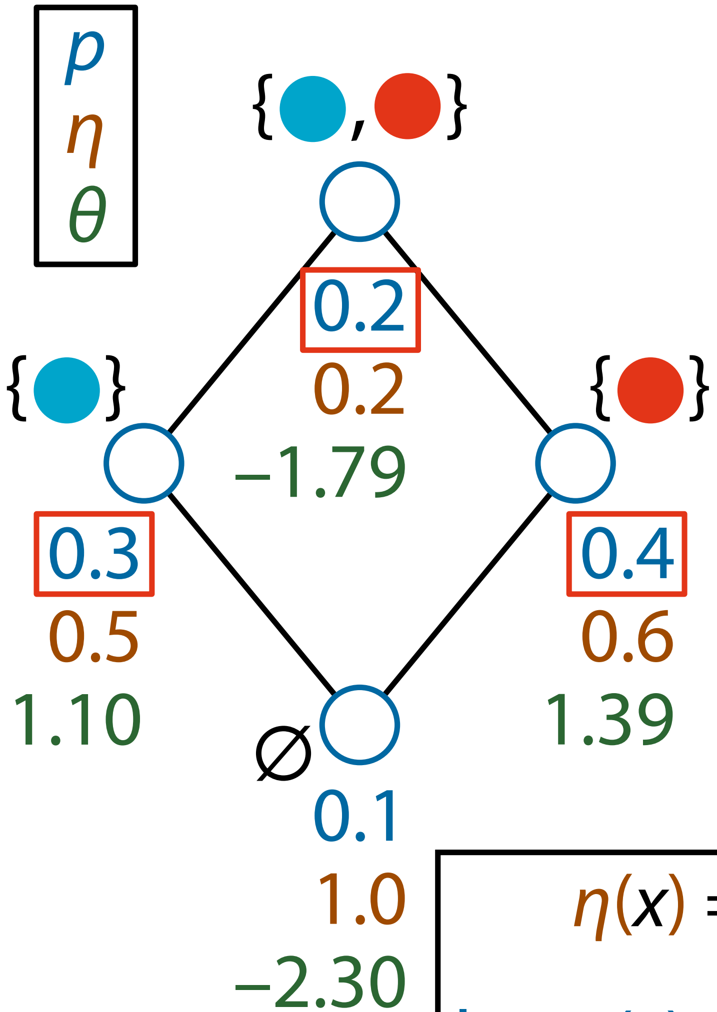
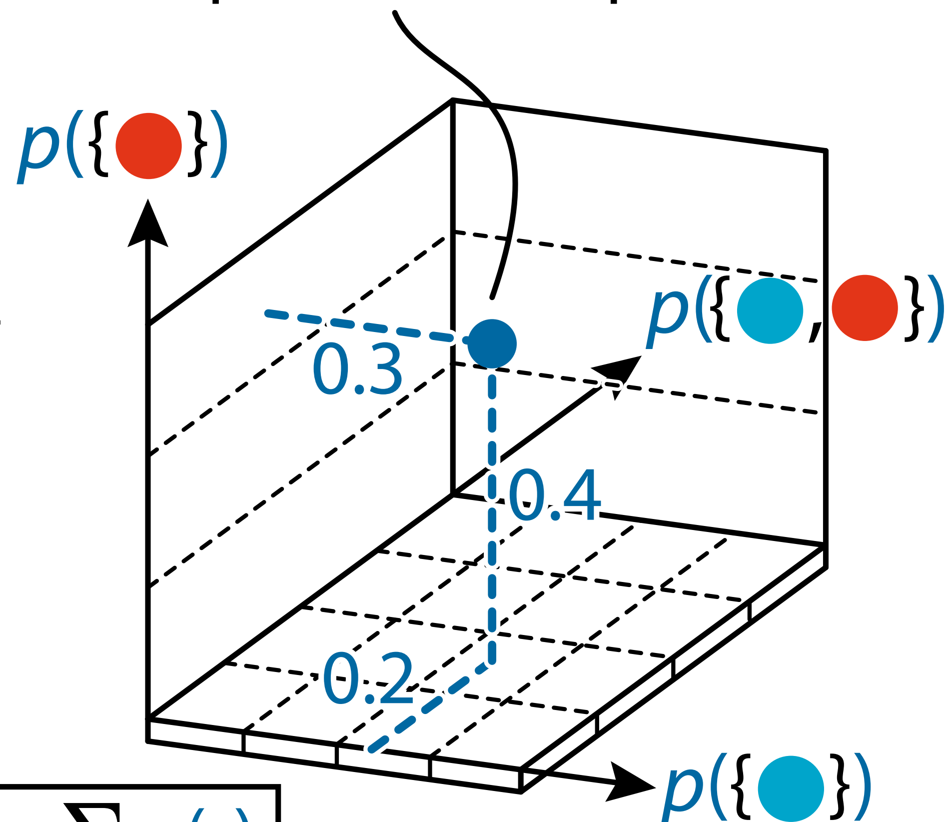
Triple for each node



Probability distribution is a “point” in 3D space



$$\eta(x) = \sum_{s \geq x} p(s)$$

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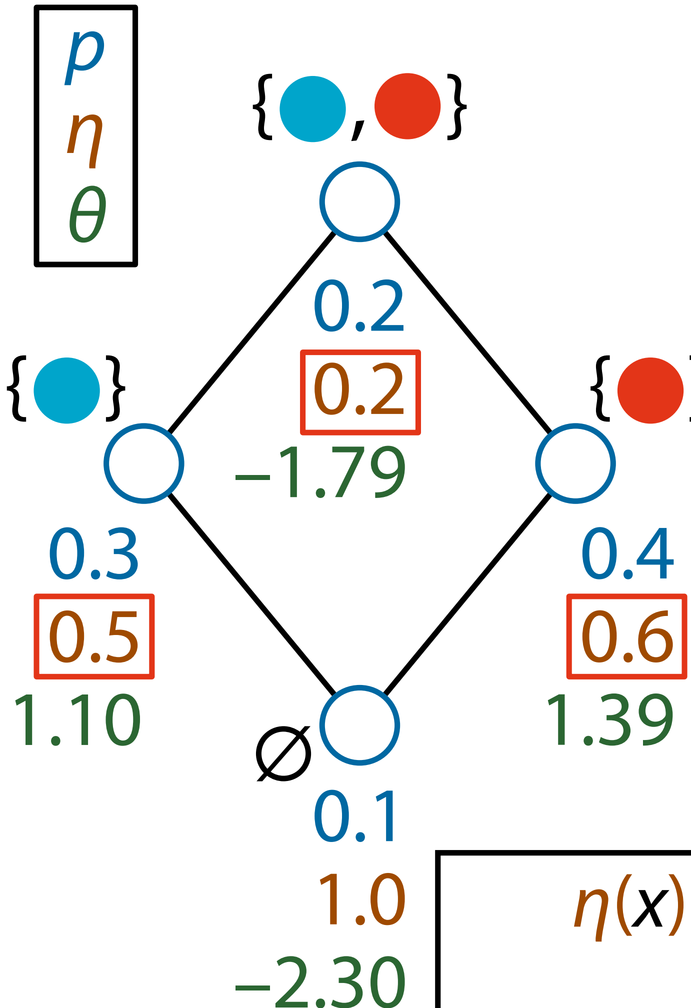
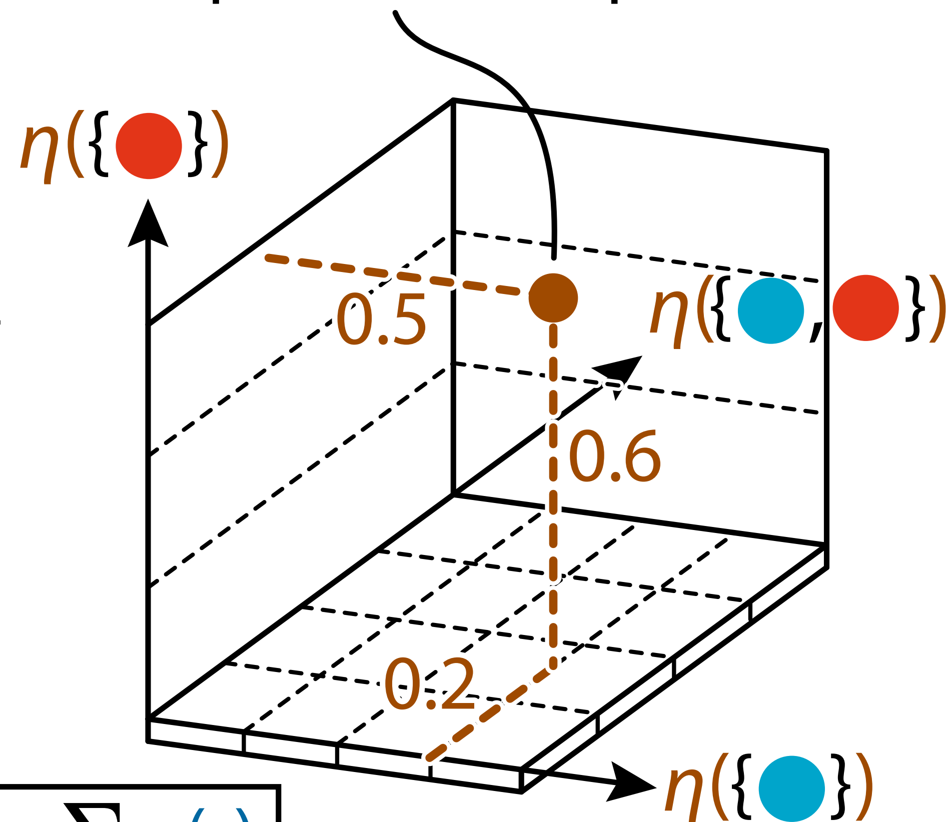
Triple for each node



Probability distribution is a "point" in 3D space



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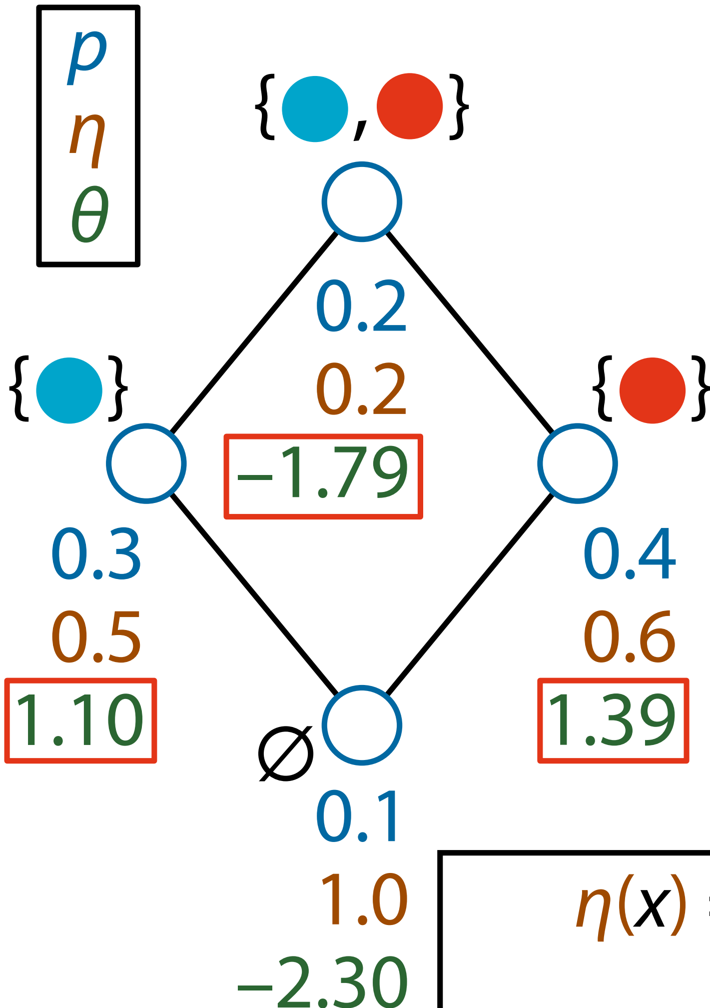
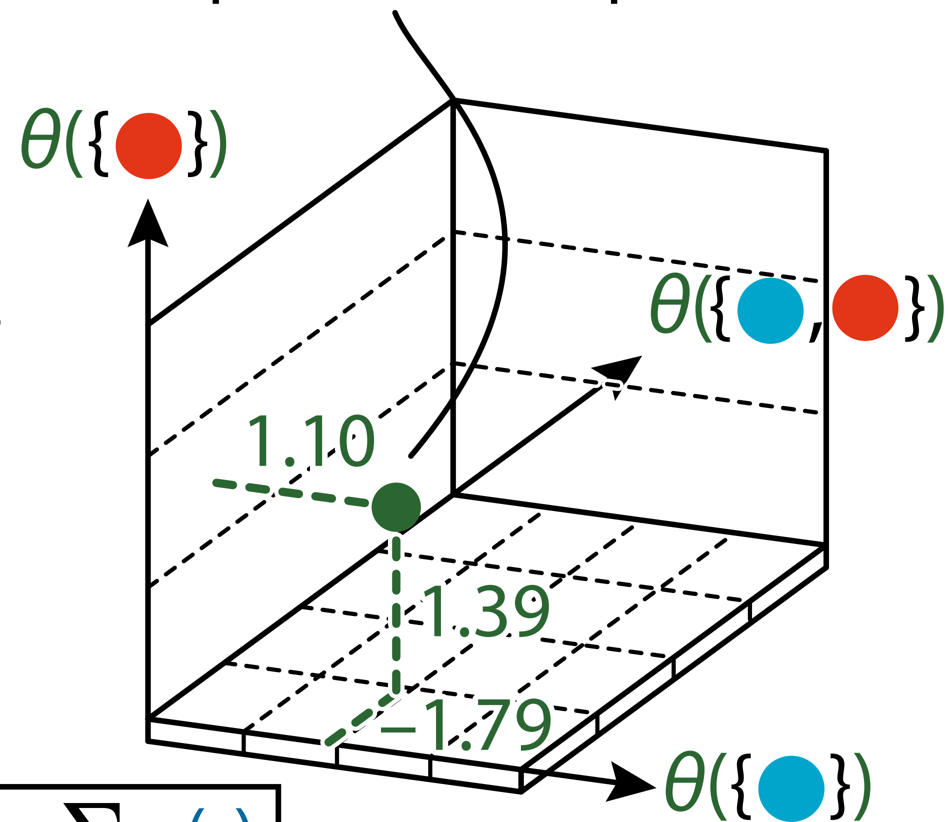
Triple for each node



Probability distribution is a "point" in 3D space



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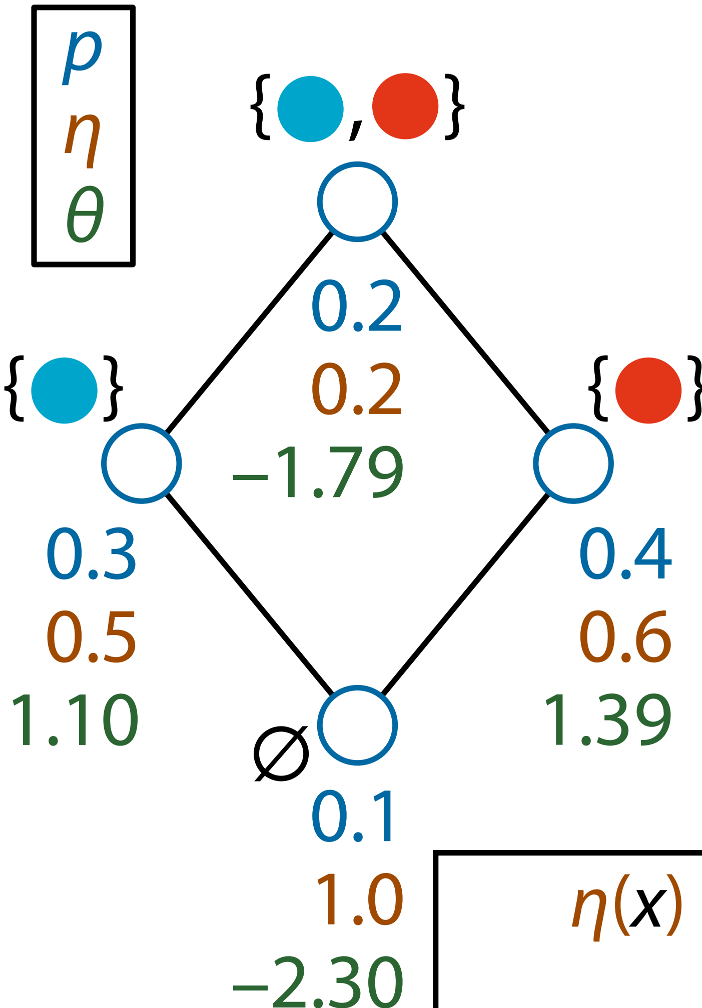
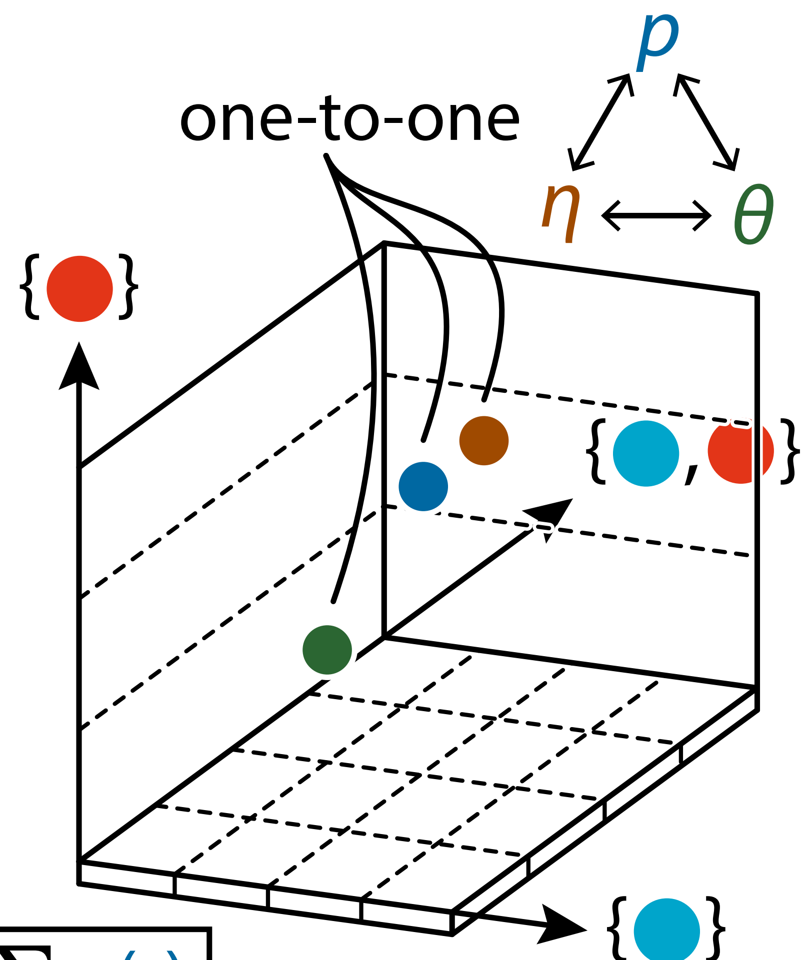
Triple for each node





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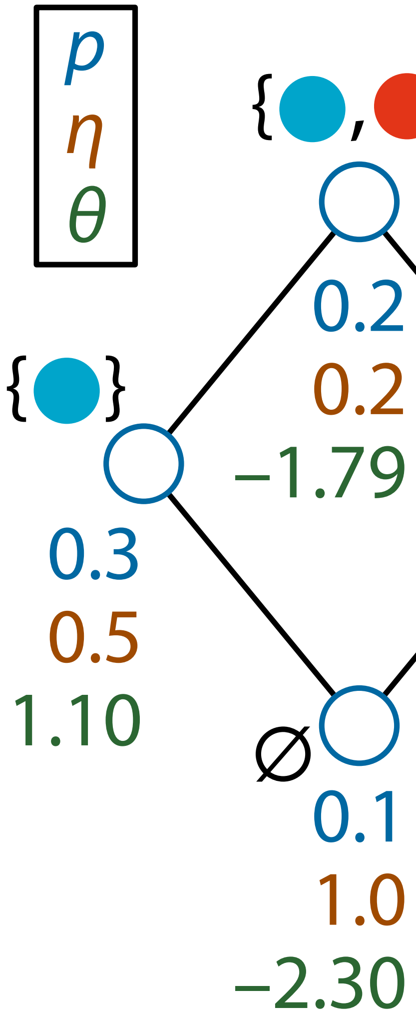
Triple for each node



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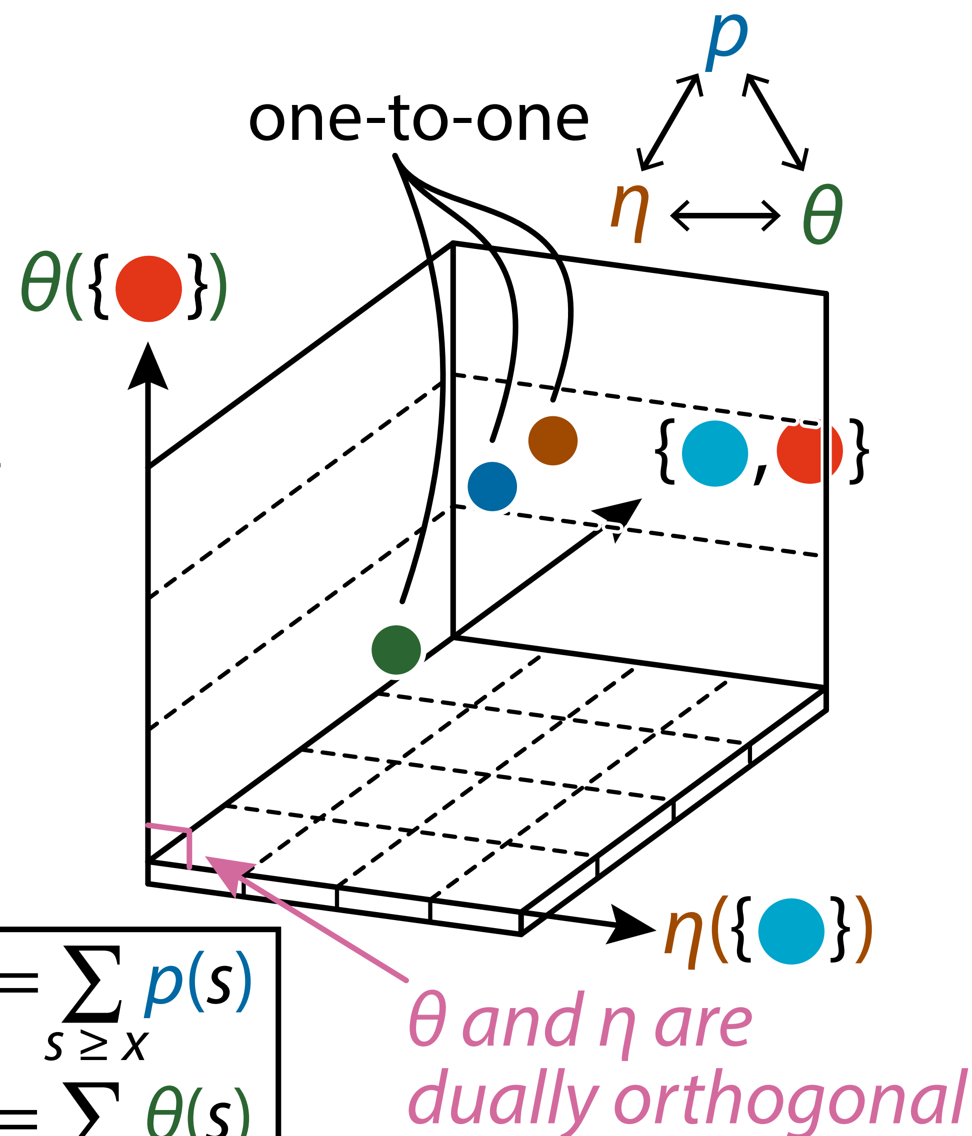
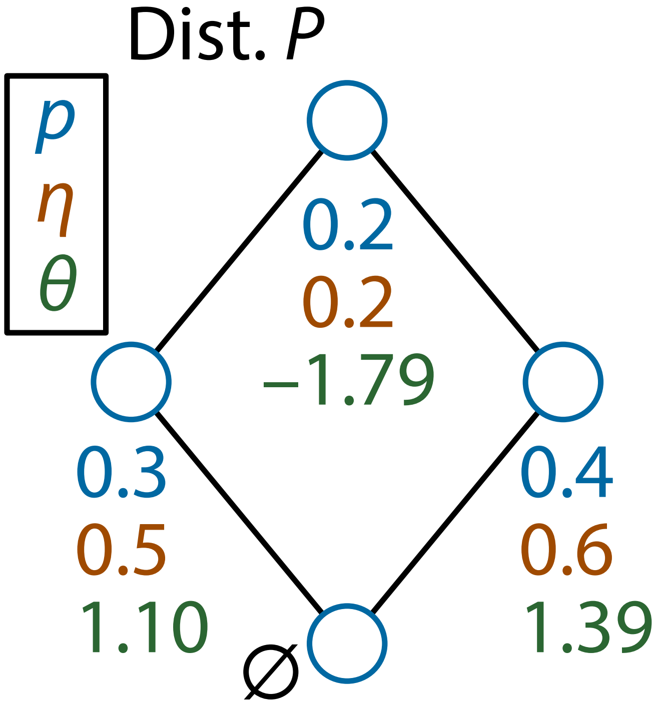
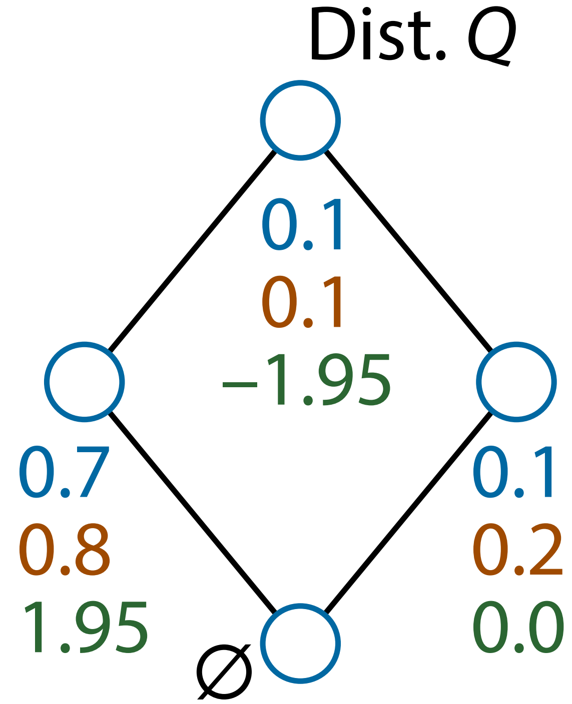
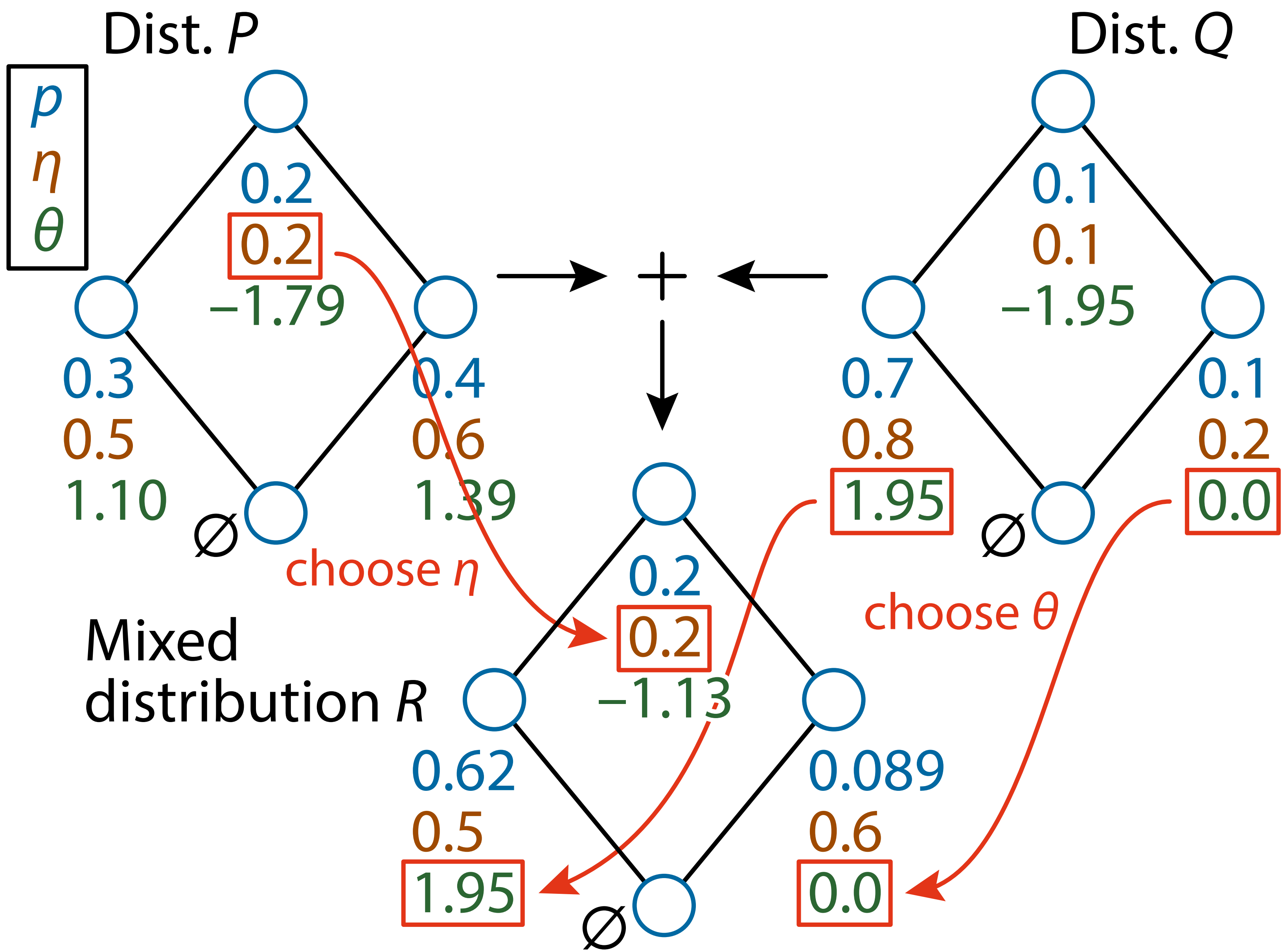
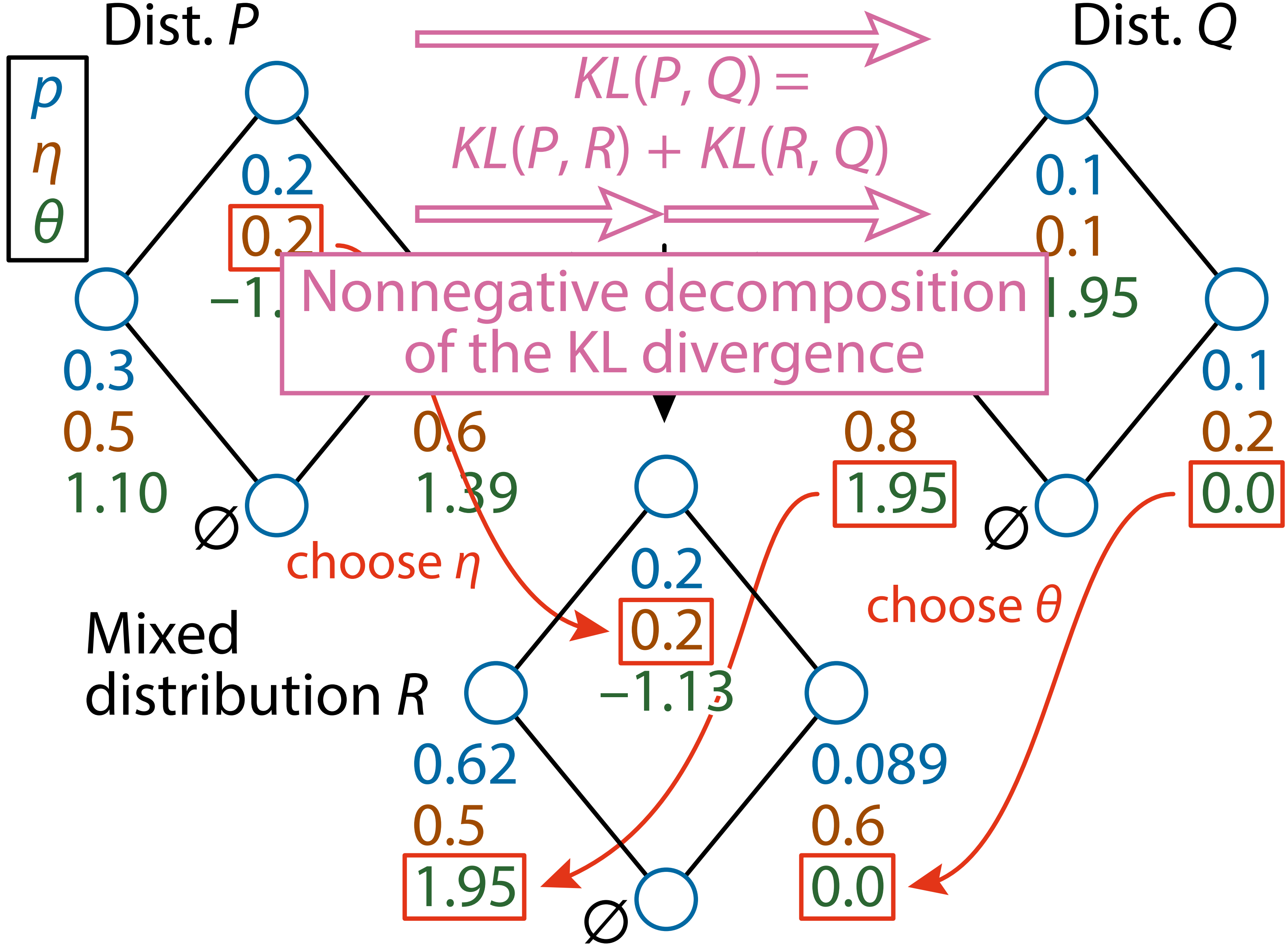


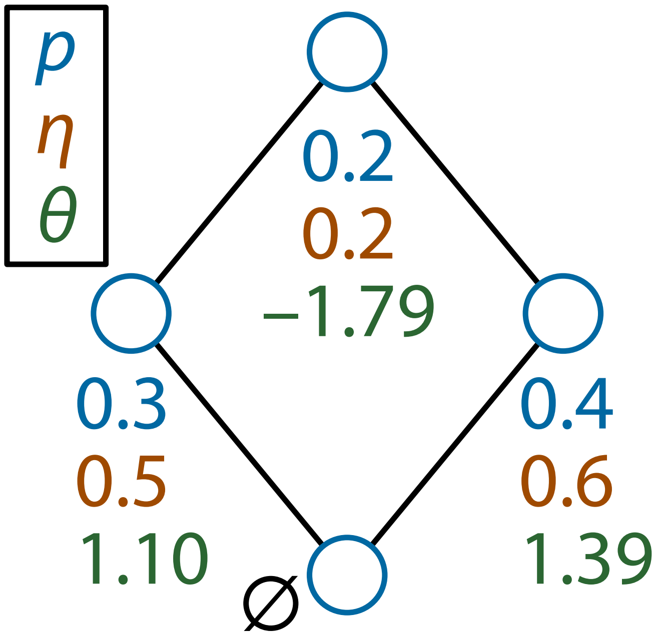

Dist. P
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅


Dist. Q
0.1
0.1
−1.95
0.7
0.8
1.95
0.1
0.2
0.0
∅

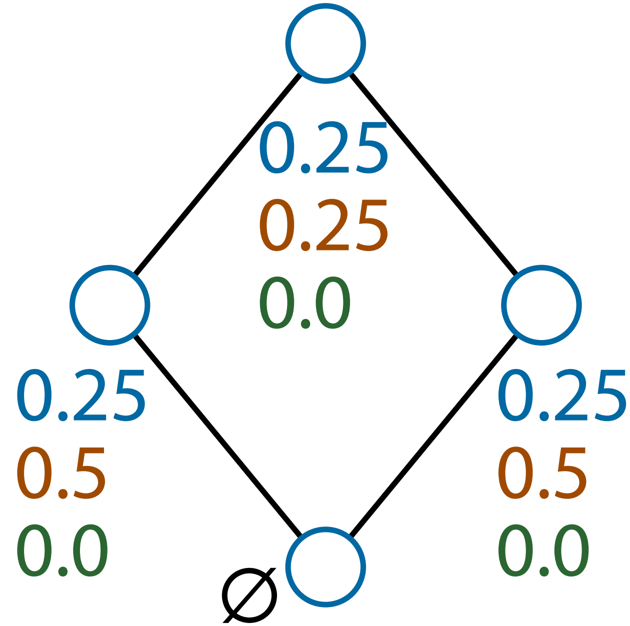

Dist. P
Dist. Q
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
0.1
0.1
−1.95
0.7
0.8
1.95
0.1
0.2
0.0
∅
+
choose η
choose θ
Mixed
distribution R
0.2
0.2
−1.13
0.62
0.5
1.95
0.089
0.6
0.0
∅

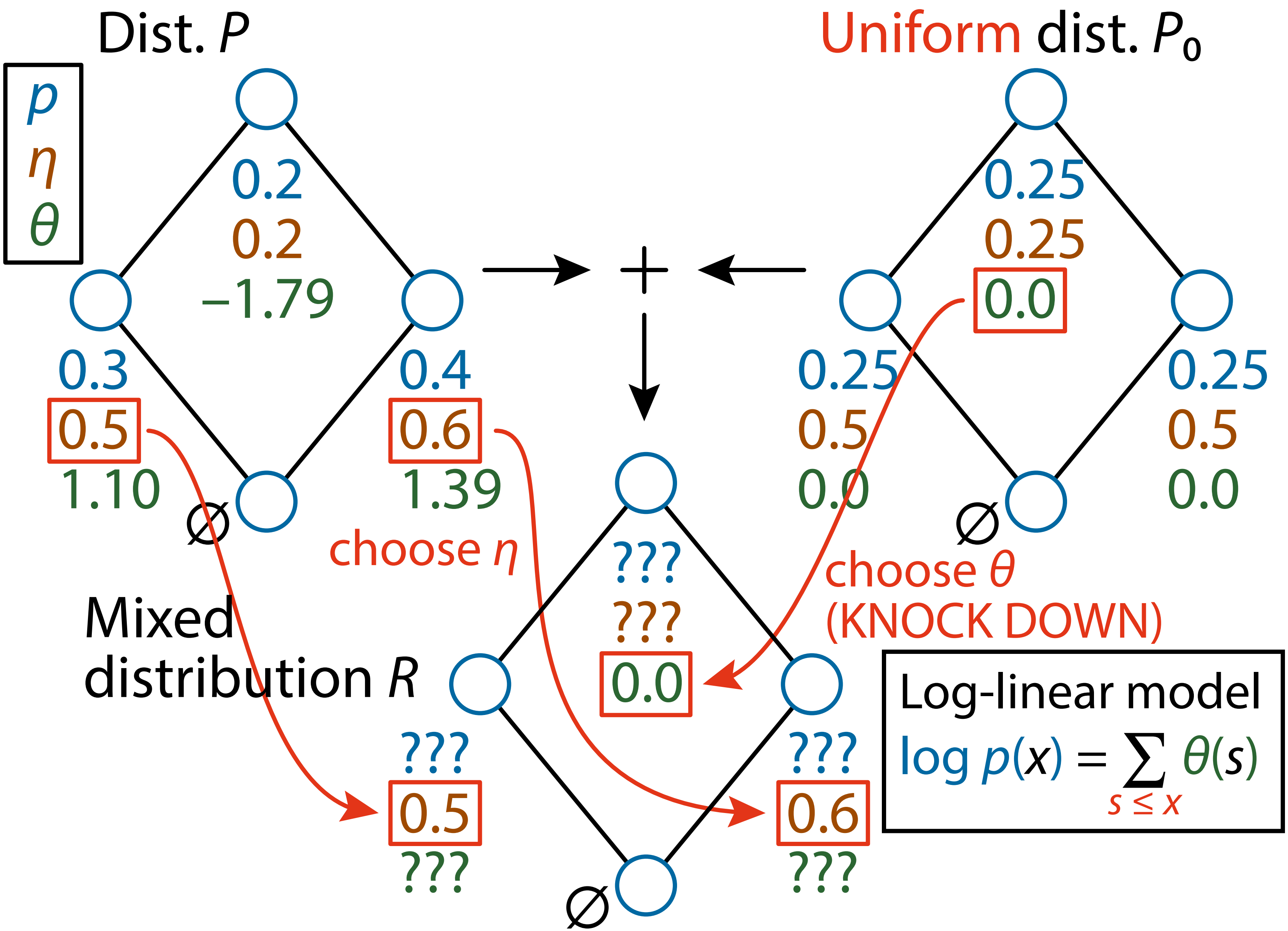

Dist. P
Dist. Q
p
η
θ
KL(P, Q) =
KL(P, R) + KL(R, Q)
Nonnegative decomposition
of the KL divergence
0.2
0.2
−1.
0.3
0.5
1.10
0.6
1.39
0.1
0.1
1.95
0.1
0.2
0.0
0.8
1.95
choose η
choose θ
Mixed
distribution R
0.2
0.2
−1.13
0.62
0.5
1.95
0.089
0.6
0.0
∅
∅
∅

Dist. P

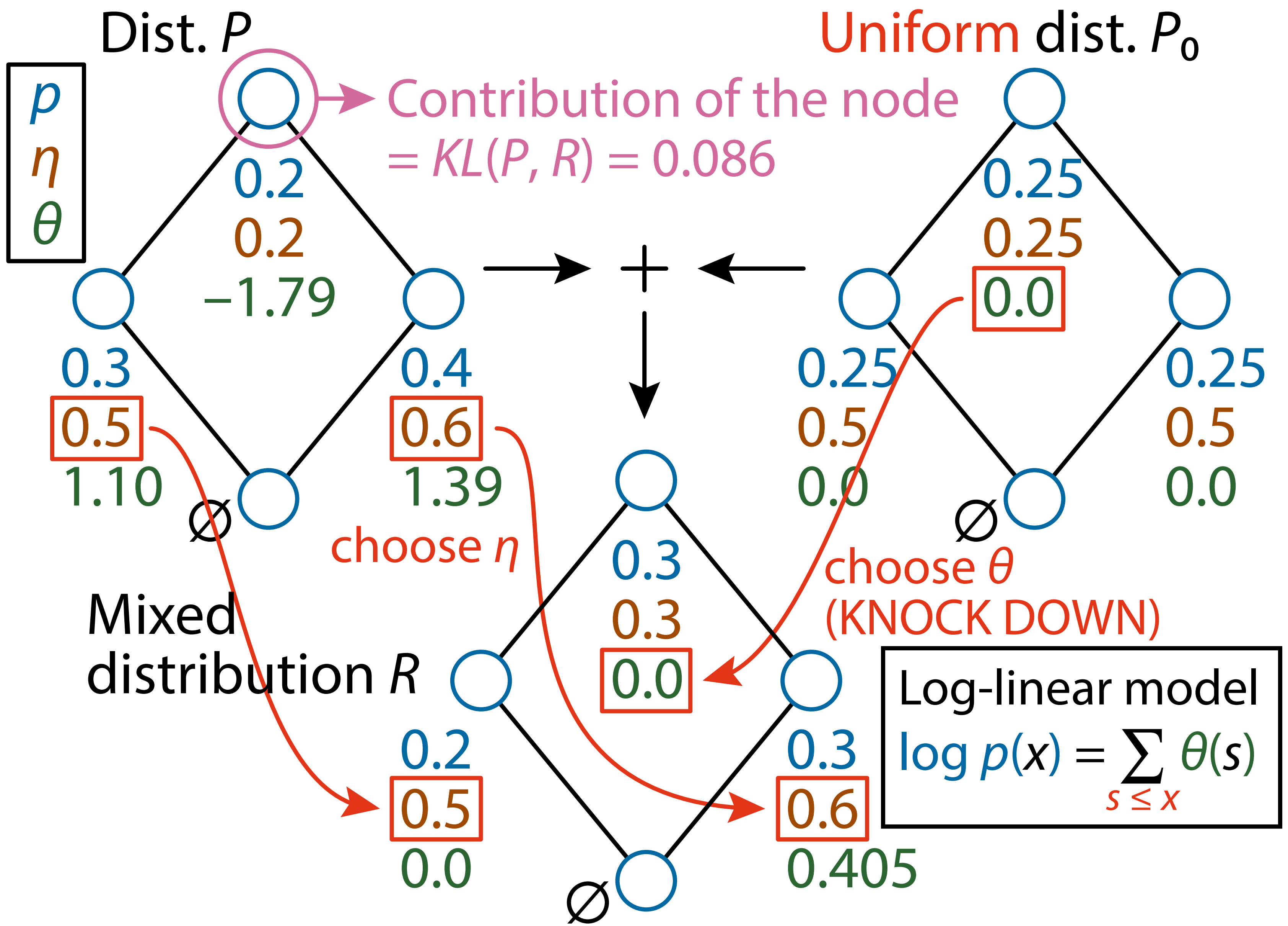


Uniform dist. P_0

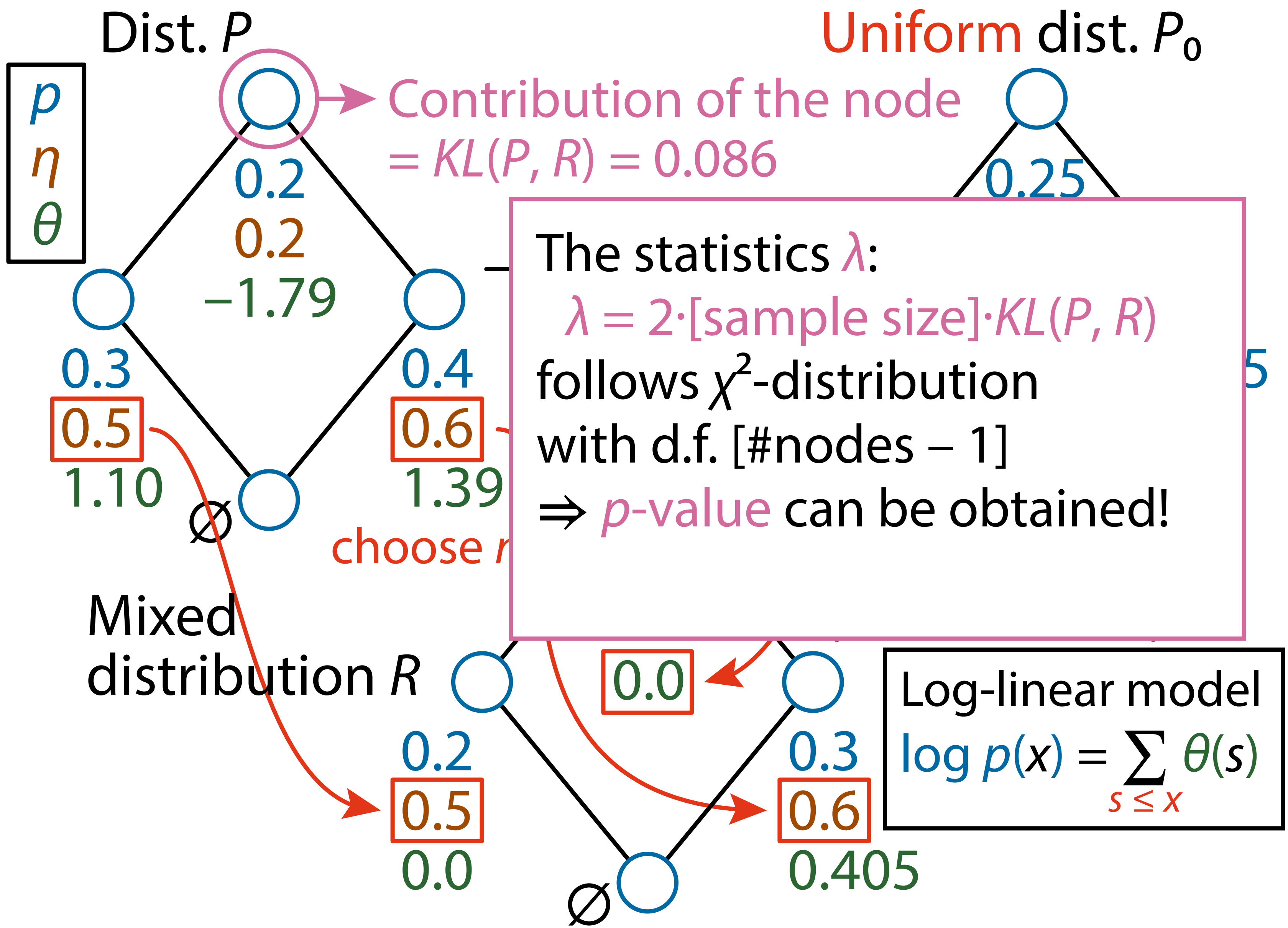




Dist. P
p
η
θ
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
choose η
Uniform dist. P_0
0.25
0.25
0.0
0.25
0.5
0.0
0.25
0.5
0.0
∅
choose θ
(KNOCK DOWN)
Mixed distribution R
???
???
0.0
???
0.5
???
???
0.6
???
∅
Log-linear model
$\log p(x) = \sum_{s \le x} \theta(s)$



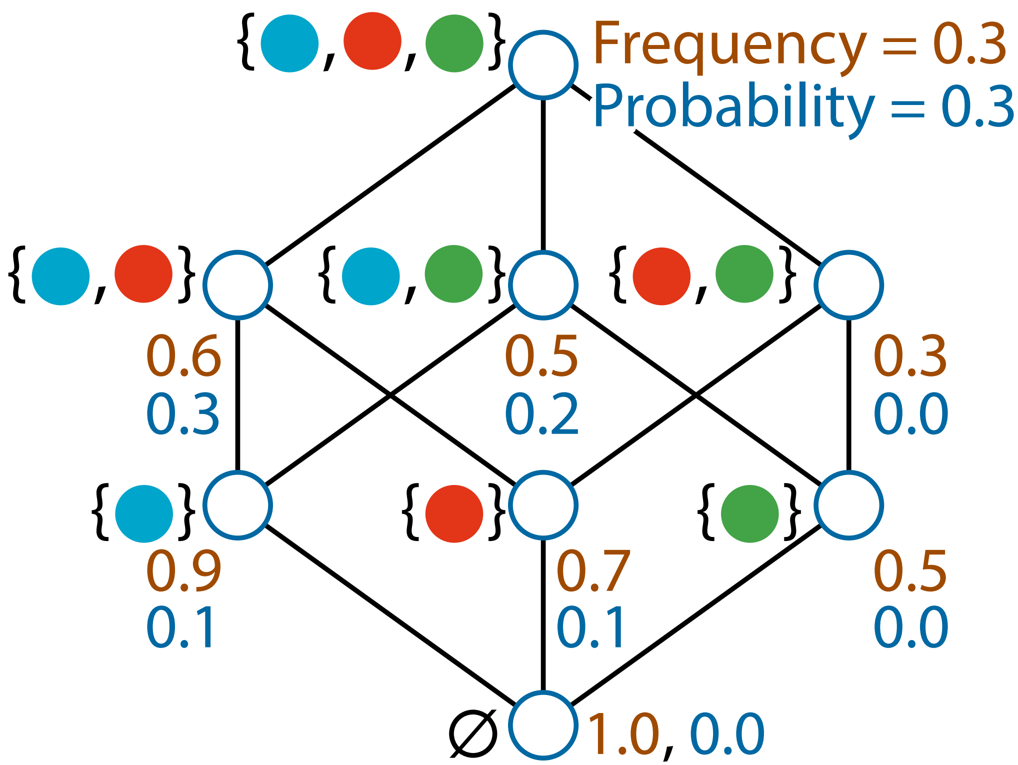
Dist. P
p
η
θ
Contribution of the node
$= KL(P, R) = 0.086$
Uniform dist. P_0
0.2
0.2
−1.79
0.3
0.5
1.10
0.4
0.6
1.39
∅
0.25
0.25
0.0
0.25
0.5
0.0
0.25
0.5
0.0
∅
choose η
choose θ
(KNOCK DOWN)
Mixed
distribution R
0.3
0.3
0.0
0.2
0.5
0.0
0.3
0.6
0.405
∅
Log-linear model
$\log p(x) = \sum_{s \le x} \theta(s)$


Dist. P
Uniform dist. P0
p
η
θ
Contribution of the node
= KL(P, R) = 0.086
0.2
0.2
−1.79
0.25
0.3
0.5
1.10
0.4
0.6
1.39
choose r
The statistics λ:
λ = 2·[sample size]·KL(P, R)
follows χ2-distribution
with d.f. [#nodes – 1]
⇒ p-value can be obtained!
Mixed
distribution R
0.0
0.2
0.5
0.0
0.3
0.6
0.405
Log-linear model
log p(x) = ∑ θ(s)
s ≤ x

Make a Poset from Data

Dataset

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0

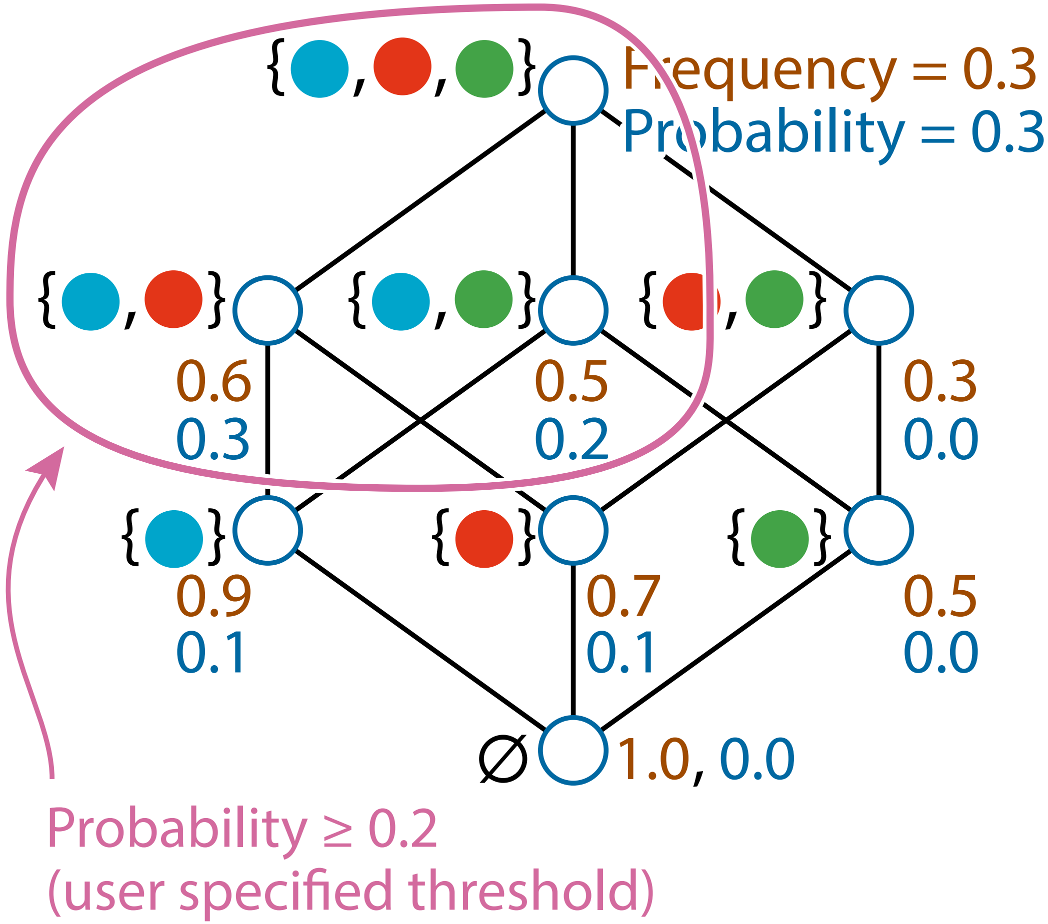


Number of nodes = $2^{\#\text{features}}$
⇒combinatorial explosion!

Make a Poset from Data

Dataset

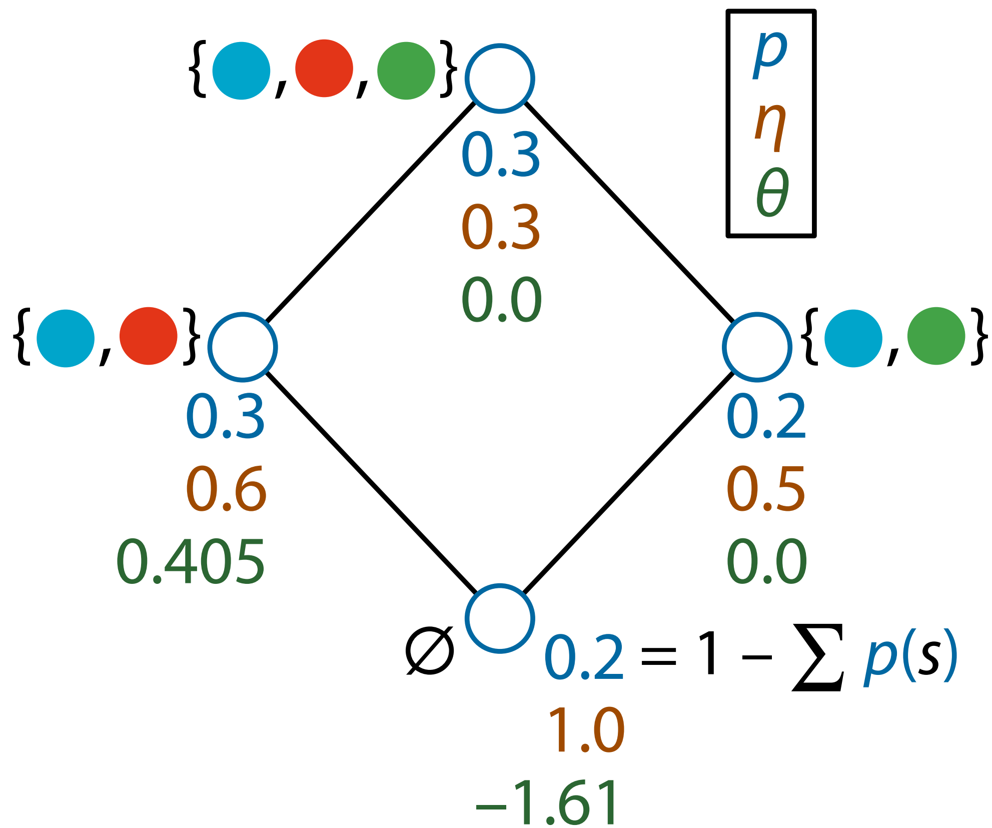
ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0



Remove Nodes with Probability 0

Dataset

ID 1:	1	1	0
ID 2:	1	1	1
ID 3:	1	1	0
ID 4:	1	1	1
ID 5:	1	1	0
ID 6:	1	0	1
ID 7:	1	0	1
ID 8:	1	1	1
ID 9:	1	0	0
ID10:	0	1	0



Example on Real Data (kosarak)

features: 41,270

ID 1: 1 1 0

ID 2: 1 1 1

ID 3: 1 1 0 …

ID 4: 1 1 1

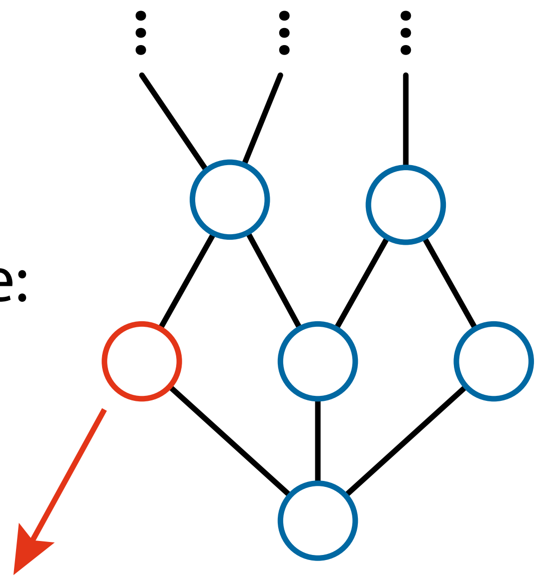
ID 5: 1 1 0

⋮ ⋮

Sample size: 990,002

Total runtime: 4.95 seconds

nodes: 3,253
(Threshold: 10^{-5})



significant interactions: 583

Single feature: 537

Pairwise interactions: 41

Triple interactions: 5

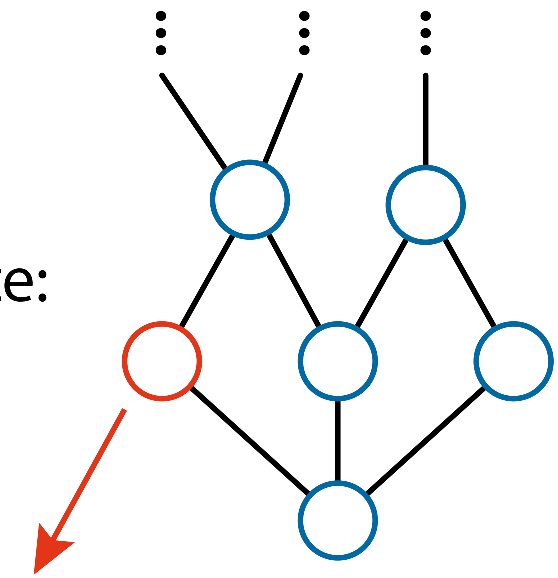
Example on Real Data (accidents)

features: 468

ID 1:	1	1	0	
ID 2:	1	1	1	
ID 3:	1	1	0	...
ID 4:	1	1	1	
ID 5:	1	1	0	
⋮		⋮		

Sample size: 340,183

nodes: 281
(Threshold: 5×10^{-6})



significant interactions: 280
features in each interaction is between 26 to 41

Total runtime: 4.95 seconds

Conclusion

- We build information geometry for posets (partially ordered sets)
  - Natural connection between the information geometric dual coordinates and the partial order structure
  - Code: `https://git.io/decomp`
- We can decompose a probability distribution and asses the significance of any-order interactions
- Related papers:
  - S. Amari, *Information geometry on hierarchy of probability distributions*, IEEE Trans. on Information Theory (2001)
  - H. Nakahara, S. Amari, *Information-geometric measure for neural spikes*, Neural Computation (2002)