Legendre Decomposition for Tensors

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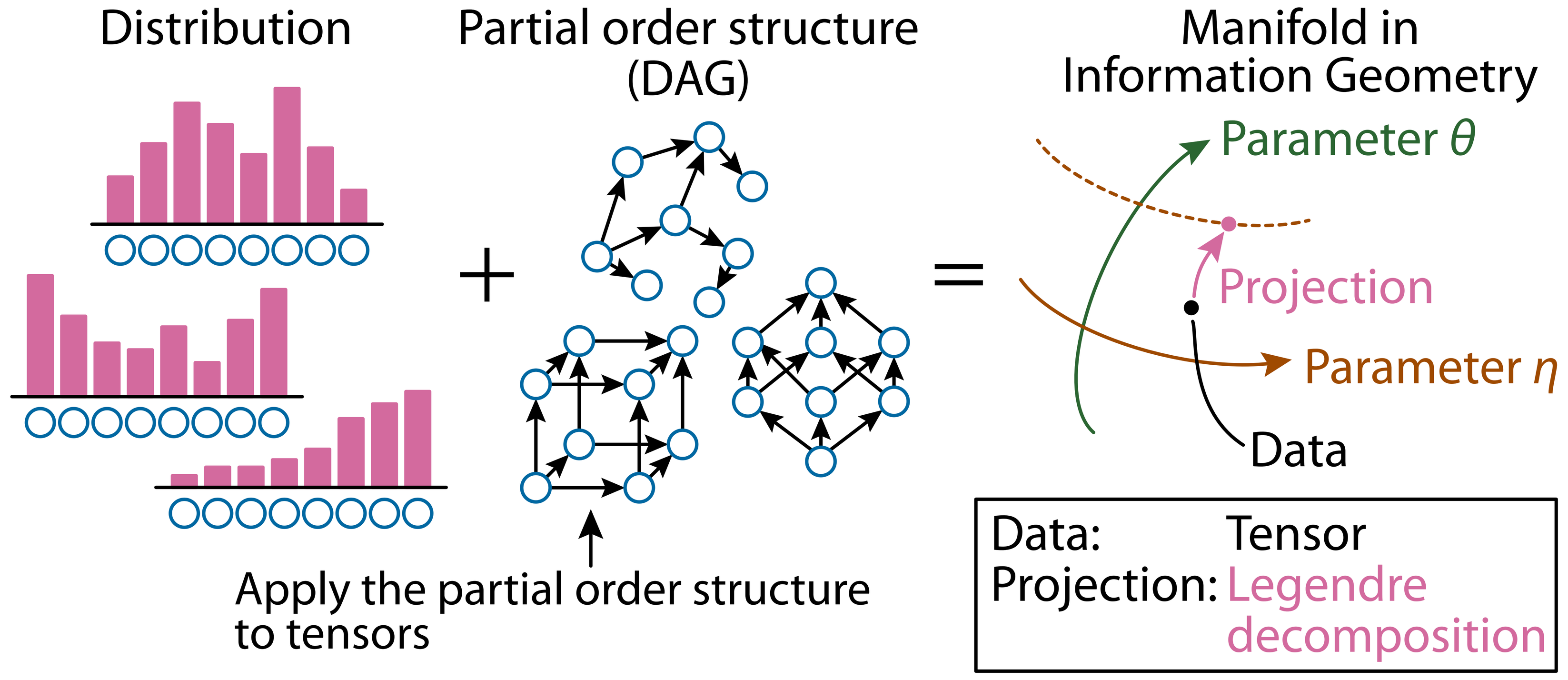
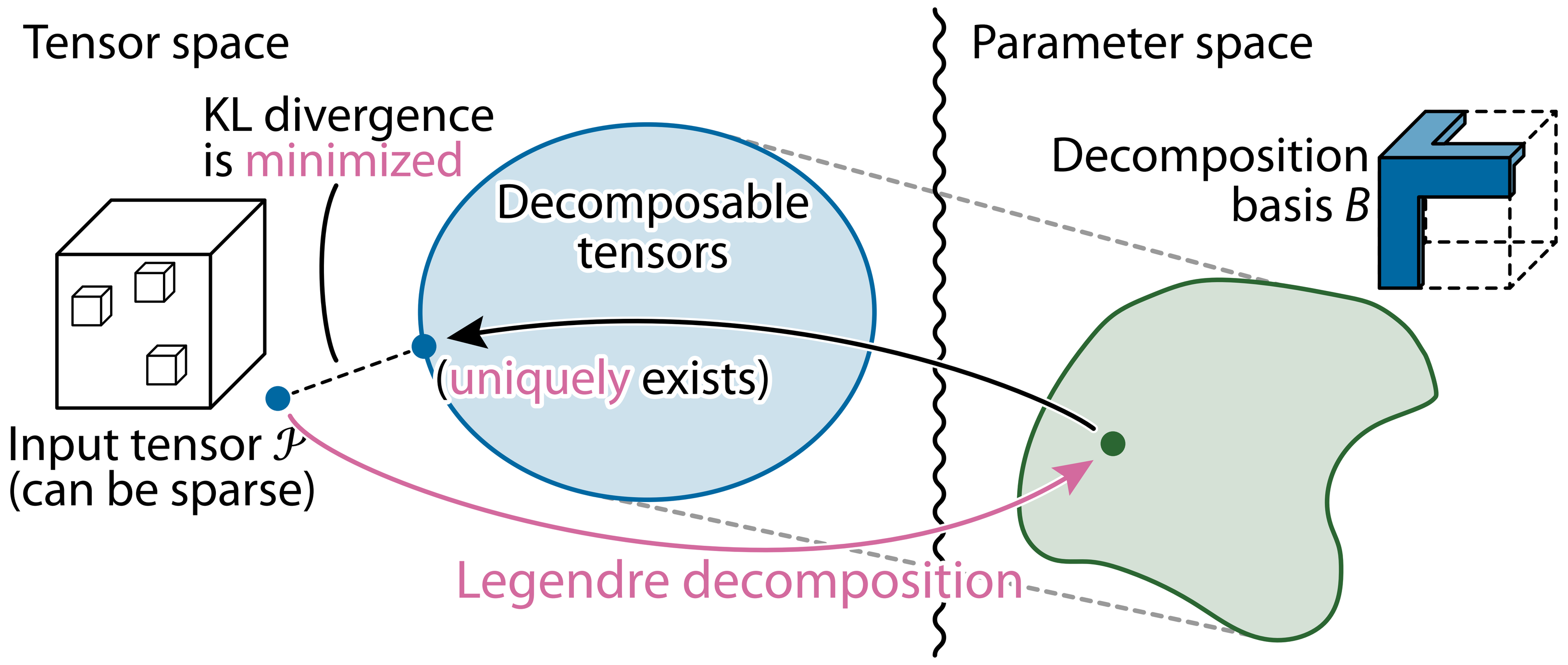
Our Approach



Summary

- We present Legendre decomposition for tensors
  - A new nonnegative decomposition method
  - A tensor is factorized into a multiplicative combination of parameters
- Our proposal is theoretically supported by information geometry
  - The reconstructed tensor is unique and always minimizes the KL divergence from an input tensor



Properties of Legendre Decomposition

- Given $\mathcal{P} \in \mathbb{R}_{\geq 0}^{I_1 \times I_2 \times \dots \times I_N}$, Legendre decomposition finds $\mathcal{Q}$, where
  (i) $\mathcal{Q}$ always exists, (ii) $\mathcal{Q}$ is unique, and
  (iii) $\mathcal{Q}$ is the best approximation in the sense of the KL divergence:

$$\mathcal{Q} = \mathrm{argmin}_{\mathcal{R} \in \boldsymbol{\mathcal{S}}_B} D_{\mathrm{KL}}(\mathcal{P}, \mathcal{R}),$$

$$\boldsymbol{\mathcal{S}}_B = \left\{ \mathcal{R} \in \mathbb{R}_{\geq 0}^{I_1 \times I_2 \times \dots \times I_N} \;\middle|\; \mathcal{R} \text{ is fully decomposable with } B \right\}$$

Legendre Decomposition

Input matrix:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

Legendre decomposition →

Reconstructed matrix:

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix}$$

↕ θ: parameter

Decomposition basis (arbitrary subset of indices)

$$\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & 0 \\ \theta_{31} & \theta_{32} & 0 \end{bmatrix} \qquad \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & 0 & 0 \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & 0 & 0 \end{bmatrix}$$

$$q_{ij} = \prod_{k \leq i, l \leq j} \exp(\theta_{kl})$$

↕ η: constraint

$$\begin{bmatrix} q_{22} & q_{23} & q_{24} \\ q_{32} & q_{33} & q_{34} \\ q_{42} & q_{43} & q_{44} \end{bmatrix} \qquad \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & & \\ \eta_{31} & \eta_{32} & & \\ \eta_{41} & \eta_{42} & & \end{bmatrix}$$

$$\eta_{ij} = \sum_{k \geq i, l \geq j} q_{kl}$$

Input matrix:

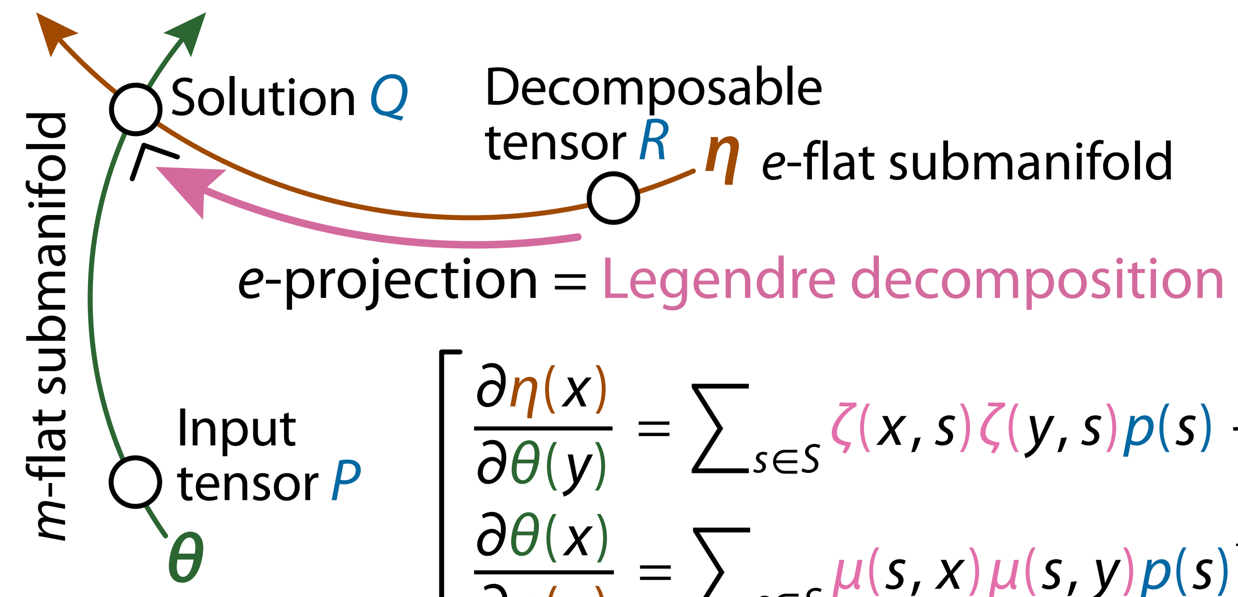
$$\begin{bmatrix} \hat{\eta}_{11} & \hat{\eta}_{12} & \hat{\eta}_{13} & \hat{\eta}_{14} \\ \hat{\eta}_{21} & \hat{\eta}_{22} & & \\ \hat{\eta}_{31} & \hat{\eta}_{32} & & \\ \hat{\eta}_{41} & \hat{\eta}_{42} & & \end{bmatrix}$$

Legendre decomposition →
Find θ s.t. $\hat{\eta} = \eta$

Reconstructed matrix:

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & & \\ \eta_{31} & \eta_{32} & & \\ \eta_{41} & \eta_{42} & & \end{bmatrix}$$

Information Geometry

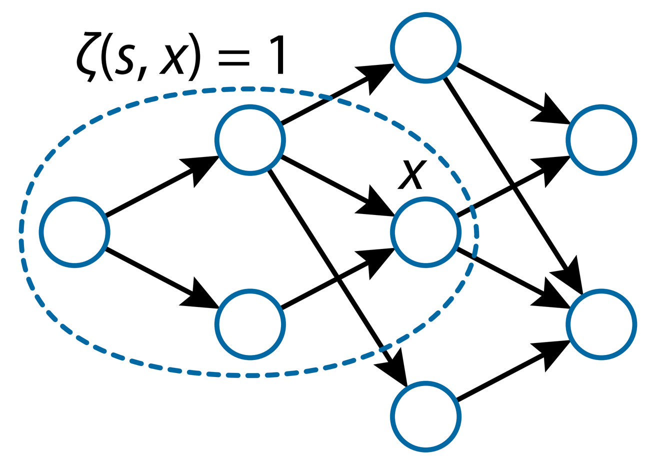


e-projection = Legendre decomposition (= MLE)

$$\frac{\partial \eta(x)}{\partial \theta(y)} = \sum_{s \in S} \zeta(x, s)\zeta(y, s)p(s) - \eta(x)\eta(y)$$

$$\frac{\partial \theta(x)}{\partial \eta(y)} = \sum_{s \in S} \mu(s, x)\mu(s, y)p(s)^{-1}$$

Capture partial order structure



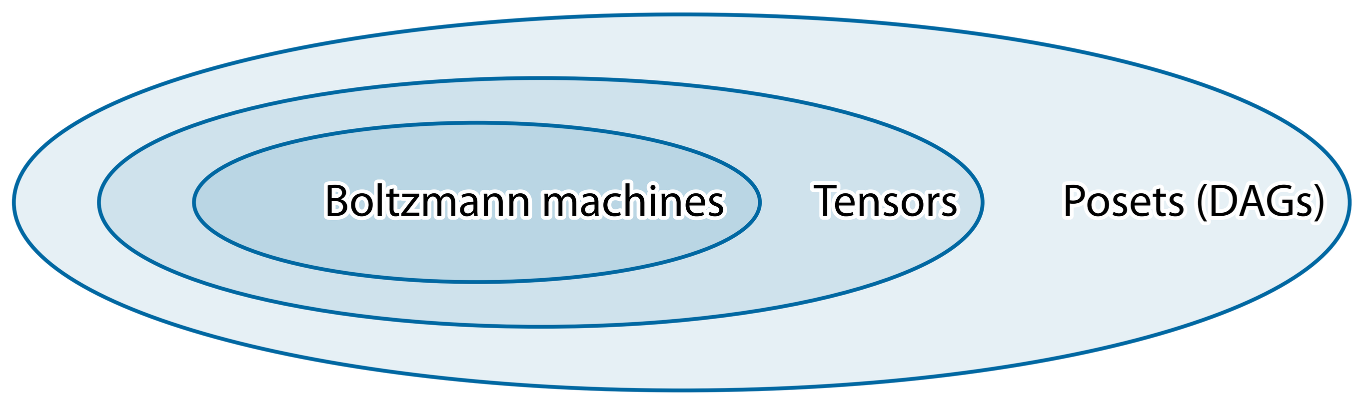
- Zeta function $\zeta : S \times S \to \{0, 1\}$

$$\zeta(s, x) = \begin{cases} 1 & \text{if } s \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

- Möbius function $\mu : S \times S \to \mathbb{Z}$

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq s < y} \mu(x, s) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

  - We have $\zeta\mu = I$, that is; $\sum_{s \in S} \zeta(s, y)\mu(x, s) = \sum_{x \leq s \leq y} \mu(x, s) = \delta_{xy}$



Experiments on MNIST